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On finite free Fisher information for eigenvectors of a modular operator ☆

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ABSTRACT

Suppose M is a von Neumann algebra equipped with a faithful normal state φ and generated by a finite set $G = G^*$, $|G| \geq 2$. We show that if G consists of eigenvectors of the modular operator Δ_φ with finite free Fisher information, then the centralizer M^φ is a II_1 factor and M is either a type II_1 factor or a type III_λ factor, $0 < \lambda \leq 1$, depending on the eigenvalues of G. Furthermore, $(M^\varphi)' \cap M = \mathbb{C}$, M^φ does not have property Γ, and M is full provided it is type III_λ, $0 < \lambda < 1$.

0. Introduction

Given random variables $x_1, \ldots, x_n$ in a non-commutative probability space (M, φ), it is natural to ask what information about the distribution of a polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ can be gleaned from the distributions of $x_1, \ldots, x_n$. If $p = x_1 + x_2$ or $p = x_1 x_2$ with x_1 freely independent from x_2, the theory of free additive and multi-

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plicative convolutions tells us everything about the distribution of p, but (until recently) without the strict *regularity condition* of free independence little could be deduced about the distribution of a general polynomial.

Shlyakhtenko and Skoufranis studied the distributions of matrices of polynomials in freely independent random variables $x_1, \ldots, x_n$ and their adjoints, and in particular showed that if $x_1, \ldots, x_n$ were semicircular random variables, then any self-adjoint polynomial has diffuse spectrum [21]. Mai, Speicher, and Weber later improved upon this result by showing that if $x_1, \ldots, x_n$ are self-adjoint random variables, not necessarily freely independent or having semicircular distributions but instead having finite free Fisher information, then $x_1, \ldots, x_n$ are algebraically free, any non-constant self-adjoint polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ has diffuse spectrum, and $W^*(x_1, \ldots, x_n)$ contains no zero divisors for $\mathbb{C}[x_1, \ldots, x_n]$ [15]. Charlesworth and Shlyakhtenko further improved on this result by weakening the assumption of finite free Fisher information to having full free entropy dimension, and showed that under stronger assumptions on $x_1, \ldots, x_n$ one can assert that the spectral measure of $p \in \mathbb{C}[x_1, \ldots, x_n]$ is non-singular [3]. These techniques have since been applied by Hartglass to show that certain elements in C^*-algebras associated to weighted graphs have diffuse spectrum [14]. In this paper, these techniques are brought to bear on non-tracial von Neumann algebras.

We consider a von Neumann algebra M with a faithful normal state φ, and a finite generating set G. We will further assume that G has finite free Fisher information with respect to the state φ, and that each $y \in G$ is an "eigenoperator"; that is, scaled by the modular automorphism group: $\sigma_t^\varphi(y) = \lambda_y^{it} y$ for some $\lambda_y > 0$. Under these assumptions, we obtain a criterion for when polynomials $\mathbb{C}[G]$ in the centralizer M^φ have diffuse spectrum (*cf.* Corollary 5.10). Our context is inspired by Shlyakhtenko's free Araki–Woods factors, which are non-tracial von Neumann algebras generated by *generalized circular elements* (operators scaled by the action of the modular automorphism group, *cf.* [18, Section 4]).

Regularity conditions on $x_1, \ldots, x_n$ can also have consequences on the von Neumann algebra generated by these operators. Indeed, Dabrowski [12] showed that if $x_1, \ldots, x_n$ in a tracial non-commutative probability space have finite free Fisher information, then these operators generate a factor without property Γ. The non-tracial analogue of this result, which considers the centralizer M^φ as well as M, is the content of the two main results of this paper. The first is concerned with factoriality:

Theorem A. *Let M be a von Neumann algebra with a faithful normal state φ. Suppose M is generated by a finite set $G = G^*$, $|G| \geq 2$, of eigenoperators of σ^φ with finite free Fisher information. Then $(M^\varphi)' \cap M = \mathbb{C}$. In particular, M^φ is a II_1 factor and if $H < \mathbb{R}_+^\times$ is the closed subgroup generated by the eigenvalues of G then*

$$M \text{ is a factor of type} \begin{cases} \mathrm{III}_1 & \text{if } H = \mathbb{R}_+^\times \\ \mathrm{III}_\lambda & \text{if } H = \lambda^{\mathbb{Z}},\ 0 < \lambda < 1 \\ \mathrm{II}_1 & \text{if } H = \{1\}. \end{cases}$$