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Quantum differentiability of essentially bounded functions on Euclidean space

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We investigate the properties of the singular values of the quantised derivatives of essentially bounded functions on $\mathbb{R}^d$ with $d > 1$. The commutator $i[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]$ of an essentially bounded function f on $\mathbb{R}^d$ acting by pointwise multiplication on $L^2(\mathbb{R}^d)$ and the sign of the Dirac operator $\mathcal{D}$ acting on $\mathbb{C}^{2^{\lfloor d/2 \rfloor}} \otimes L^2(\mathbb{R}^d)$ is called the quantised derivative of f. We prove the condition that the function $x \mapsto \|(\nabla f)(x)\|_2^d := ((\partial_1 f)(x)^2 + \ldots + (\partial_d f)(x)^2)^{d/2}$, $x \in \mathbb{R}^d$, being integrable is necessary and sufficient for the quantised derivative of f to belong to the weak Schatten d-class. This problem has been previously studied by Rochberg and Semmes, and is also explored in a paper of Connes, Sullivan and Telemann. Here we give new and complete proofs using the methods of double operator integrals. Furthermore, we prove a formula for the Dixmier trace of the d-th power of the absolute value of the quantised derivative. For real valued f, when $x \mapsto \|(\nabla f)(x)\|_2^d$ is integrable, there exists a constant $c_d > 0$ such that for every continuous normalised trace φ on the weak trace class $\mathcal{L}_{1,\infty}$ we have $\varphi(|[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]|^d) = c_d \int_{\mathbb{R}^d} \|(\nabla f)(x)\|_2^d dx$.

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1. Introduction

Let $d > 1$ be an integer, and let $x_1, x_2, \ldots, x_d$ be the coordinates of $\mathbb{R}^d$. Given a separable Hilbert space H, we denote the algebra of all bounded linear operators on H by $\mathcal{L}_\infty(H)$. The singular value function of a bounded operator $A \in \mathcal{L}_\infty(H)$ is defined by

$$\mu(t, A) = \inf\{\|A(1-P)\| : P \text{ is a finite rank projection}, \operatorname{Tr}(P) \le t\},\ t \ge 0.$$

The sequence $\{\mu(n, A)\}_{n=0}^\infty$ is called the sequence of singular values of the operator A. When A is a compact operator then $\mu(n, A)$, $n \ge 0$, is the $(n+1)$-th eigenvalue of the absolute value $|A|$ when the sequence of eigenvalues is arranged in decreasing order. We define the Schatten–Von Neumann space $\mathcal{L}_p(H)$, $p \in (0, \infty]$, as the subspace of operators in $\mathcal{L}_\infty(H)$ with a sequence of singular values in ℓ^p. Similarly the Schatten–Lorentz space $\mathcal{L}_{p,q}(H)$ is defined as the operators with singular values in $\ell^{p,q}$, for $p, q \in (0, \infty]$. When $p \ne \infty$ an operator $A \in \mathcal{L}_{p,q}(H)$ is compact. See [8, Chapter 4] for details on these spaces. We will suppress the dependence on H and write $\mathcal{L}_{p,q}$ when the Hilbert space is clear from context.

Given $A \in \mathcal{L}_{p,q}$ with a sequence of singular values $\{\mu(n, A)\}_{n=0}^\infty$, the quasinorm $\|A\|_{p,q}$ is defined to be the $\ell^{p,q}$ norm of $\{\mu(n, A)\}_{n=0}^\infty$.

For $j = 1, \ldots, d$, we define D_j to be the derivative in the direction x_j,

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} = -i\partial_j.$$

When $f \in L^\infty(\mathbb{R}^d)$ is not a smooth function then $D_j f$ denotes the distributional derivative of f. We also consider D_j as a self-adjoint operator on $L^2(\mathbb{R}^d)$ with its standard domain of square integrable functions with a square integrable weak derivative in the direction x_j. This is equivalent to the closure of the symmetric operator D_j restricted to Schwartz functions. We use the notation $\nabla f = i(D_1 f, D_2 f, \ldots, D_d f)$ for an essentially bounded function $f \in L^\infty(\mathbb{R}^d)$. For a square integrable function f with a square integrable derivative in each direction we consider ∇ as an unbounded operator from $L^2(\mathbb{R}^d)$ to the Bochner space $L^2(\mathbb{R}^d, \mathbb{C}^d)$.

Let $N = 2^{\lfloor d/2 \rfloor}$. We use d-dimensional Euclidean gamma matrices, which are $N \times N$ self-adjoint complex matrices $\gamma_1, \ldots, \gamma_d$ satisfying the anticommutation relation,

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k},\quad 1 \le j, k \le d,$$

where δ is the Kronecker delta. The precise choice of matrices satisfying this relation is unimportant so we assume that a choice is fixed for the rest of this paper.

Using this choice of gamma matrices, we can define the d-dimensional Dirac operator,

$$\mathcal{D} = \sum_{j=1}^d \gamma_j \otimes D_j.$$

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