Journal of Functional Analysis 273 (2017) 2488–2520

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Sample variance in free probability

Wiktor Ejsmont [a], Franz Lehner [b,*]

[a] *Wydział Matematyki i Informatyki, Uniwersytet Im. Adama Mickiewicza, Collegium Mathematicum, Umultowska 87, 61-614 Poznań, Poland and Department of Mathematics and Cybernetics, Wrocław University of Economics, ul. Komandorska 118/12, 53-345 Wrocław, Poland*

[b] *Institut für Diskrete Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria*

A R T I C L E I N F O

Article history:
Received 2 August 2016
Accepted 31 May 2017
Available online 17 June 2017
Communicated by D. Voiculescu

Dedicated to our friend and mentor Marek Bożejko on the occasion of his 70-th birthday

MSC:
primary 46L54
secondary 62E10

Keywords:
Sample variance
Free infinite divisibility
Cancellation of free cumulants
Wigner semicircle law

A B S T R A C T

Let $X_1, X_2, \dots, X_n$ denote i.i.d. centered standard normal random variables, then the law of the sample variance $Q_n = \sum_{i=1}^n (X_i - \overline{X})^2$ is the χ^2-distribution with $n-1$ degrees of freedom. It is an open problem in classical probability to characterize all distributions with this property and in particular, whether it characterizes the normal law. In this paper we present a solution of the free analogue of this question and show that the only distributions, whose free sample variance is distributed according to a free χ^2-distribution, are the semicircle law and more generally so-called *odd* laws, by which we mean laws with vanishing higher order even cumulants. In the way of proof we derive an explicit formula for the free cumulants of Q_n which shows that indeed the odd cumulants do not contribute and which exhibits an interesting connection to the concept of R-cyclicity.

* Corresponding author.
E-mail addresses: wiktor.ejsmont@gmail.com (W. Ejsmont), lehner@math.tugraz.at (F. Lehner).

http://dx.doi.org/10.1016/j.jfa.2017.05.007
0022-1236/

1. Introduction

Many questions in classical statistics involve characterization problems, which usually are instances of the following very general question:

Problem 1.1. Let $X_1, X_2, \ldots, X_n$ be independent random variables with common unknown distribution function F, and $T := T(X_1, X_2, \ldots, X_n)$ a statistic, based on $X_1, X_2, \ldots, X_n$, with distribution function G. Can F be recovered from G?

Problems of this kind are the central leitmotiv of the fundamental work of Kagan, Linnik and Rao [17]. In the present paper we solve the free version of the following problem, which is still open in classical probability and might be called χ^2-conjecture, see [17, p. 466]:

Conjecture 1.2. *If $X_1, X_2, \ldots, X_n$ are non-degenerate, independently and identically distributed classical random variables with finite non-zero variance σ^2, then a necessary and sufficient condition for X_1 to be normal is that $\sum_{i=1}^n (X_i - \overline{X})^2/\sigma^2$ be distributed as classical chi-square distribution with $n-1$ degrees of freedom.*

The classical χ^2-conjecture was studied previously by several authors. The first result is due to Ruben [28], who proved the conjecture under the assumption that either $n = 2$ or X_1 is symmetric. It is not known whether the symmetry hypothesis can be dropped for $n \geq 3$. In a later paper [29] Ruben used combinatorial tools to show that the symmetry condition can be dropped provided the sum of squares of the sample observations about the sample mean, divided by σ^2, is distributed as chi-square for two distinct sample sizes $m \neq n$ and $m, n \geq 2$. The proof given by Ruben is based on the cumulants of the sample variance and is somewhat complicated. Shortly later a simpler and more direct proof based on the moments of the sample variance was presented by Bondesson [5].

The original problem was solved recently by Golikova and Kruglov [13] under the additional assumption that $X_1, X_2, \ldots, X_n$ are independent infinitely divisible random variables.

The following related characterization problem was solved by Kagan and Letac [18]: Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables and assume that the distribution of the quadratic statistic $\sum_{i=1}^n (X_i - \overline{X} + a_i)^2$ depends only on $\sum_{i=1}^n a_i^2$. Then each X_i have distribution $N(0, \sigma)$.

In the present paper we answer analogous questions in free probability. Free probability and free convolution was introduced by Voiculescu in [37] as a tool to study the von Neumann algebras of free groups. Free probability is now an established field of research with deep connections to combinatorics, random matrix theory, representation theory and many analogies to classical probability. Let us restrict our discussion to two specific ones, which are relevant to the problems discussed in the present paper. On the analytic side the Bercovici–Pata bijection [4] provides a one-to-one correspondence between infinitely divisible measures with respect to classical and free convolution. For