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On the operator homology of the Fourier algebra and its *cb*-multiplier completion

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A B S T R A C T

We study various operator homological properties of the Fourier algebra $A(G)$ of a locally compact group G. Establishing the converse of two results of Ruan and Xu [35], we show that $A(G)$ is relatively operator 1-projective if and only if G is IN, and that $A(G)$ is relatively operator 1-flat if and only if G is inner amenable. We also exhibit the first known class of groups for which $A(G)$ is not relatively operator C-flat for any $C \geq 1$. As applications of our techniques, we establish a hereditary property of inner amenability, answer an open question of Lau and Paterson [24], and answer an open question of Anantharaman-Delaroche [1] on the equivalence of inner amenability and Property (W). In the bimodule setting, we show that relative operator 1-biflatness of $A(G)$ is equivalent to the existence of a contractive approximate indicator for the diagonal G_Δ in the Fourier–Stieltjes algebra $B(G \times G)$, thereby establishing the converse to a result of Aristov, Runde, and Spronk [3]. We conjecture that relative 1-biflatness of $A(G)$ is equivalent to the existence of a quasi-central bounded approximate identity in $L^1(G)$, that is, G is QSIN, and verify the conjecture in many special cases. We finish with an application to the operator homology of $A_{cb}(G)$,

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giving examples of weakly amenable groups for which $A_{cb}(G)$ is not operator amenable.

1. Introduction

The operator homology of the Fourier algebra $A(G)$ of a locally compact group G has been a topic of interest in abstract harmonic analysis since Ruan's seminal work [34], where, among other things, he established the equivalence of amenability of G and operator amenability of $A(G)$. From the perspective of Pontryagin duality, this result is the dual analogue of Johnson's celebrated equivalence of amenability of G and (operator) amenability of $L^1(G)$ [22]. In much the same spirit, dual analogues of various homological properties of $L^1(G)$ were established within the category of operator $A(G)$-modules, including the operator weak amenability of $A(G)$ [37], and the equivalence of discreteness of G and relative operator biprojectivity of $A(G)$ [2,43].

Continuing in this spirit, Ruan and Xu (implicity) showed that $A(G)$ is relatively operator 1-projective whenever G is an IN group (see also [17]), and that $A(G)$ is relatively operator 1-flat whenever G is inner amenable [35]. In this paper, we establish the converse of both of these results, and exhibit the first known class of groups – including every connected non-amenable group – for which $A(G)$ is not relatively operator C-flat for any $C \geq 1$. Along the way, we show that inner amenability passes to closed subgroups, answer an open question of Lau and Paterson [24], and answer an open question of Anantharaman-Delaroche [1, Problem 9.1] on the equivalence of inner amenability and Property (W).

The relative operator biflatness of $A(G)$ has been studied by Ruan and Xu [35] and Aristov, Runde, and Spronk [3], where it was shown (by different methods) that $A(G)$ is relatively operator biflat whenever G is QSIN, meaning $L^1(G)$ has a quasi-central bounded approximate identity (see [3,27,38]). The approach of Aristov, Runde, and Spronk is via approximate indicators, where they show that $A(G)$ is relatively operator C-biflat whenever the diagonal subgroup $G_\Delta \leq G \times G$ has a bounded approximate indicator in $B(G \times G)$ of norm at most C. One of the main results of this paper establishes the converse when $C = 1$, that is, $A(G)$ is relatively operator 1-biflat if and only if G_Δ has a contractive approximate indicator in $B(G \times G)$. Recalling that $A(G)$ is operator amenable precisely when $A(G \times G)$ has a bounded approximate diagonal [34], we see that $A(G)$ is relatively operator 1-biflat precisely when $A(G\times G)$ has a contractive approximate diagonal in the *Fourier–Stieltjes algebra* $B(G\times G)$, a result which elucidates the relationship between operator amenability and relative operator biflatness for $A(G)$, and for completely contractive Banach algebras more generally.

We conjecture that relative operator 1-biflatness of $A(G)$ is equivalent to the QSIN condition, and we verify the conjecture in many special cases. For a discrete group H