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Boundary behavior of nonlocal minimal surfaces ☆

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We consider the behavior of the nonlocal minimal surfaces in the vicinity of the boundary. By a series of detailed examples, we show that nonlocal minimal surfaces may stick at the boundary of the domain, even when the domain is smooth and convex. This is a purely nonlocal phenomenon, and it is in sharp contrast with the boundary properties of the classical minimal surfaces.

In particular, we show stickiness phenomena to half-balls when the datum outside the ball is a small half-ring and to the side of a two-dimensional box when the oscillation between the datum on the right and on the left is large enough.

When the fractional parameter is small, the sticking effects may become more and more evident. Moreover, we show that

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lines in the plane are unstable at the boundary: namely, small compactly supported perturbations of lines cause the minimizers in a slab to stick at the boundary, by a quantity that is proportional to a power of the perturbation.

In all the examples, we present concrete estimates on the stickiness phenomena. Also, we construct a family of compactly supported barriers which can have independent interest.

1. Introduction

It is well known (see e.g. [19,17]) that the classical minimal surfaces do not stick at the boundary. Namely, if Ω is a convex domain and E is a set that minimizes the perimeter among its competitors in Ω, then ∂E is transverse to $\partial\Omega$ at their intersection points.

In this paper we show that the situation for the nonlocal minimal surfaces is completely different. Indeed, we prove that nonlocal interactions can favor stickiness at the boundary for minimizers of a fractional perimeter.

The mathematical framework in which we work was introduced in [7] and is the following. Given $s \in (0, 1/2)$ and an open set $\Omega \subseteq \mathbb{R}^n$, we define the s-perimeter of a set $E \subseteq \mathbb{R}^n$ in Ω as

$$\mathrm{Per}_s(E, \Omega) := L(E \cap \Omega, E^c) + L(\Omega \setminus E, E \setminus \Omega),$$

where $E^c := \mathbb{R}^n \setminus E$ and, for any disjoint sets F and G, we use the notation

$$L(F, G) := \iint_{F\times G} \frac{dx\, dy}{|x - y|^{n+2s}}.$$

We say that E is s-minimal in Ω if $\mathrm{Per}_s(E, \Omega) < +\infty$ and $\mathrm{Per}_s(E, \Omega) \leqslant \mathrm{Per}_s(F, \Omega)$ among all the sets F which coincide with E outside Ω.

With a slight abuse of language, when Ω is unbounded, we say that E is s-minimal in Ω if it is s-minimal in any bounded open subsets of Ω (for a more precise distinction between s-minimal sets and locally s-minimal sets see e.g. [21]).

Problems related to the s-perimeter naturally arise in several fields, such as the motion by nonlocal mean curvature and the nonlocal Allen–Cahn equation, see e.g. [8,25]. Also, the s-perimeter can be seen as a fractional interpolation between the classical perimeter (corresponding to the case $s \to 1/2$) and the Lebesgue measure (corresponding to the case $s \to 0$), see e.g. [22,3,9,1,14].