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Coarse and uniform embeddings

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A B S T R A C T

In these notes, we study the relation between uniform and coarse embeddings between Banach spaces as well as uniform and coarse equivalences. In order to understand these relations better, we look at the problem of when a coarse embedding can be assumed to be also topological. Among other results, we show that if a Banach space X uniformly embeds into a minimal Banach space Y, then X simultaneously coarsely and uniformly embeds into Y, and if a Banach space X coarsely embeds into a minimal Banach space Y, then X simultaneously coarsely and homeomorphically embeds into Y by a map with uniformly continuous inverse.

1. Introduction

The study of Banach spaces as metric spaces has recently increased significantly, and much has been done regarding the uniform and coarse theory of Banach spaces in the past two decades. Let (M, d) and (N, ∂) be metric spaces, and consider a map $f : (M, d) \to (N, \partial)$. For each $t \geq 0$, we define the *expansion modulus* of f as

$$\omega_f(t) = \sup\{\partial(f(x), f(y)) \mid d(x, y) \leq t\},$$

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and the *compression modulus* of f as

$$\rho_f(t) = \inf\{\partial(f(x), f(y)) \mid d(x,y) \geq t\}.$$

Hence, $\rho_f(d(x,y)) \leq \partial(f(x), f(y)) \leq \omega_f(d(x,y))$, for all $x, y \in M$. The map f is a uniform embedding if $\lim_{t\to 0_+} \omega_f(t) = 0$ and $\rho_f(t) > 0$, for all $t > 0$. We call f a *uniform homeomorphism* if f is a surjective uniform embedding. The map f is called *coarse* if $\omega_f(t) < \infty$, for all $t \geq 0$, and *expanding* if $\lim_{t\to\infty} \rho_f(t) = \infty$. If f is both expanding and coarse, f is called a *coarse embedding*. A coarse embedding f which is also *cobounded*, i.e., $\sup_{y\in N} \partial(y, f(M)) < \infty$, is called a *coarse equivalence* (for more details on those concepts see Subsection 2.2).

Although the uniform theory of Banach spaces has already been extensively studied in the past, only recently we have been starting to understand the coarse theory better. For example, it had been known since 1984 that there are uniformly homeomorphic separable Banach spaces which are not linearly isomorphic (see [22], Theorem 1), but it was not till 2012 that N. Kalton was able to show that there are coarsely equivalent separable Banach spaces (i.e., with Lipschitz isomorphic nets) which are not uniformly homeomorphic (see [11], Theorem 8.9). However, it is still not known whether the concepts of uniform and coarse embeddability are equivalent. Precisely, the following problem remains open.

Problem 1.1. Let X and Y be Banach spaces. Does X uniformly embed into Y if and only if X coarsely embed into Y?

In [21], N. Randrianarivony had shown that a Banach space coarsely embeds into a Hilbert space if and only if it uniformly embeds into a Hilbert space. In [10], Kalton showed that the same also holds for embeddings into ℓ_∞ (Theorem 5.3). C. Rosendal had made some improvements on the problem above by showing that if X uniformly embeds into Y, then X simultaneously uniformly and coarsely embeds into $\ell_p(Y)$, for any $p \in [1, \infty)$ (see [23], Theorem 2). In particular, if X uniformly embeds into ℓ_p, then X simultaneously coarsely and uniformly embeds into ℓ_p. On the other hand, A. Naor had recently proven that there exist separable Banach spaces X and Y, and a Lipschitz map f from a net $N \subset X$ into Y such that

$$\sup_{x\in N} \|F(x) - f(x)\| = \infty,$$

for all uniformly continuous maps $F : X \to Y$ (see [17], Remark 2). Such result suggests that it may not be true (or at least not easy to show) that X uniformly embeds into Y, given that X coarsely embeds into Y.

In these notes, we study the relation between coarse embeddings (resp. coarse equivalences) and uniform embeddings (resp. uniform homeomorphisms) between Banach spaces as well as some properties shared by those notions. We are specially interested in narrowing down the difference between those concepts, and we show that, in many