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Variational estimates for the bilinear iterated Fourier integral

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We prove pointwise variational L^p bounds for a bilinear Fourier integral operator in a large but not necessarily sharp range of exponents. This result is a joint strengthening of the corresponding bounds for the classical Carleson operator, the bilinear Hilbert transform, the variation norm Carleson operator, and the bi-Carleson operator. Terry Lyon's rough path theory allows for extension of our result to multilinear estimates. We consider our result a proof of concept for a wider array of similar estimates with possible applications to ordinary differential equations.

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1. Introduction

Consider the bilinear iterated Fourier inversion integral

$$B(f_1, f_2)(x) := \int\limits_{\xi_1<\xi_2} \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)e^{ix(\xi_1+\xi_2)}d\xi_1 d\xi_2 \ . \tag{1}$$

It is a close relative of the bilinear Hilbert transform, and as such satisfies L^p bounds as in [11–14].

Given any $r \in (0, \infty)$, let T_r denote the following stronger operator

$$\sup_{K, N_0<\cdots<N_K} \Big(\sum_{j=1}^{K} \Big| \int\limits_{N_{j-1}<\xi_1<\xi_2<N_j} \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)e^{ix(\xi_1+\xi_2)}d\xi_1 d\xi_2\Big|^{r/2}\Big)^{2/r} \quad . \tag{2}$$

Thus T_r is a *variation sum* over truncations of B, in particular it dominates both B and the bi-Carleson operator considered in [24], which essentially is the limit case $r \to \infty$ of T_r.

The main result of our paper is the following theorem:

Theorem 1.1. *Assume that $r > 2$. Then T_r is bounded from $L^{p_1} \times L^{p_2}$ to L^{p_3} provided that $1/p_3 = 1/p_1 + 1/p_2$ and*

$$\max(1, \frac{2r}{3r-4}) < p_1, p_2 \le \infty \quad , \quad \max(\frac{2}{3}, \frac{r'}{2}) < p_3 < \infty \quad . \tag{3}$$

Besides strengthening [11,14,24], Theorem 1.1 also implies a range of the L^p estimates for the variation norm Carleson theorem in [26]. Namely, the variation norm Carleson estimate can be obtained by a variant of (2) without the constraint $\xi_1 < \xi_2$, which in turn can be estimated by the sum of (2) and a symmetric version of (2).

The theory of ordinary differential equations $G' = WG$ with rough driving signals W initiated by T. Lyons [17] and developed by many, for example [18], discusses similar expressions as (2) and also higher multilinear analogues of (2). Under some mild regularity assumptions, the theory allows for bootstrapping of estimates for expressions similar to (2) *with* $r \le 3$ to higher multilinear estimates, and these estimates are then used to control the iterated integrals appearing in the multilinear expansion of the solution of the equation, and consequently one obtains estimates for the solution curve $\{G(x), x \in \mathbb{R}\}$. In particular, T. Lyons' rough path theory allows for bootstrapping of our main theorem in a certain range of exponents to multi(sub)linear estimates:

Corollary 1.2. *Let $k \ge 3$. For any $r > 0$ let $T_{k,r}$ denote*

$$\sup_{K, N_0<\cdots<N_K} \Big(\sum_{j=1}^{K} \Big| \int\limits_{N_{j-1}<\xi_1<\cdots<\xi_k<N_j} \prod_{m=1}^{k} \widehat{f}_m(\xi_m)e^{ix\xi_m}d\xi_m\Big|^{r/k}\Big)^{k/r} \quad . \tag{4}$$