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Norm-attaining functionals need not contain 2-dimensional subspaces ☆

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ABSTRACT

G. Godefroy asked whether, on any Banach space, the set of norm-attaining functionals contains a 2-dimensional linear subspace. We prove that a construction due to C.J. Read provides an example of a space which does not have this property. Read found an equivalent norm $|||\cdot|||$ on c_0 such that $(c_0, |||\cdot|||)$ contains no proximinal subspaces of codimension 2. Our result is obtained through a study of the relation between the following two sentences, in which X is a Banach space and $Y \subset X$ is a closed subspace: **(A)** Y is proximinal in X, and **(B)** $Y^\perp$ consists of norm-attaining functionals. We prove that these are equivalent if X is the space $(c_0, |||\cdot|||)$, and our main theorem then follows as a corollary to Read's result.

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1. Introduction

Given a real Banach space X, norm-attaining functionals are those $x^* \in X^*$ for which there exists $x \in S_X$ such that $x^*(x) = \|x^*\|$; the set of all norm-attaining functionals on X is denoted by $\mathrm{NA}(X)$. The main motivation for the present work is the following Problem of G. Godefroy together with a recent theorem of C.J. Read below.

Problem 1.1 *([9, Problem III.], also e.g. [2] or [1]).* Does the set $\mathrm{NA}(X)$ contain a 2-dimensional subspace for any Banach space X?

The well-known theorem of E. Bishop and R.R. Phelps [4] states that for any Banach space X, the set $\mathrm{NA}(X)$ is dense in X^*; in particular, $\mathrm{NA}(X)$ is nonempty and so it clearly contains 1-dimensional subspaces. Since then, a number of papers studying the structure of $\mathrm{NA}(X)$ from various viewpoints have been written; the ones which are interesting for us are concerned with the linear structure of $\mathrm{NA}(X)$, in particular the question whether $\mathrm{NA}(X)$ is lineable or spaceable (for definition see Section 2). For example, in [2] it is shown that if X is an Asplund space enjoying the Dunford–Pettis property, then $\mathrm{NA}(X)$ is not spaceable (e.g. $C(K)$ with K scattered (Hausdorff) compactum; however, $\mathrm{NA}(C(K))$ for K infinite is always lineable—see [1]). On the other hand, in [1] the authors prove that if the Banach space X possesses an infinite-dimensional complemented subspace with a Schauder basis, then X can be equivalently renormed so that $\mathrm{NA}(X)$ is lineable. Some articles also investigate the lineability of $X^* \setminus \mathrm{NA}(X)$, e.g. [1] or [7]. In [7] it is observed that if $\mathcal{J}$ denotes the James space, then $\mathcal{J}^* \setminus \mathrm{NA}(\mathcal{J})$ is not even 2-lineable (i.e. $(\mathcal{J}^* \setminus \mathrm{NA}(\mathcal{J})) \cup \{0\}$ contains no 2-dimensional subspace).

A related problem (again [9, Problem III.]) was to decide whether every Banach space contains a proximinal subspace of codimension 2. This problem was recently solved by C.J. Read in the next theorem. Recall that a set $F \subset X$ is said to be *proximinal* if for each $x \in X$ there is $y \in F$ with $\|x - y\| = \mathrm{dist}(x, F)$.

Theorem R *(C.J. Read [16]). Let* $|||\cdot|||$ *be the renorming of* c_0 *from Definition 1. Then* $(c_0, |||\cdot|||)$ *contains no proximinal subspaces of finite codimension* 2 *or larger.*

Our main result is to show that this space also works as a counterexample for Problem 1.1. More precisely, our Theorem 4.2 states that $\mathrm{NA}(c_0, |||\cdot|||)$ contains no 2-dimensional subspaces. This is done by showing that $(c_0, |||\cdot|||)$ satisfies implication **(B)** $\Longrightarrow$ **(A)**; one can readily see that this is sufficient. In general, if X is any Banach space and $Y \subset X$ is a closed subspace, sentences **(A)** and **(B)** are the following:

(A) Y is proximinal in X.
(B) $Y^\perp := \{x^* \in X^*;\ Y \subset \mathrm{Ker}(x^*)\} \subset \mathrm{NA}(X)$.

In view of the easy observation that a functional $x^* \in X^*$ is norm-attaining if and only if $\mathrm{Ker}(x^*)$ is proximinal, it is reasonable to ask whether sentences **(A)** and **(B)** are also