Journal of Functional Analysis 272 (2017) 929–975

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Semilinear elliptic equations with Dirichlet operator and singular nonlinearities

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A R T I C L E   I N F O

Article history:
Received 20 May 2015
Accepted 31 October 2016
Available online 8 November 2016
Communicated by M. Hairer

MSC:
35J75
60J45

Keywords:
Semilinear elliptic equations
Probabilistic potential theory
Dirichlet form
Additive functional

A B S T R A C T

In the paper we consider elliptic equations of the form $-Au = u^{-\gamma} \cdot \mu$, where A is the operator associated with a regular symmetric Dirichlet form, μ is a positive nontrivial measure and $\gamma > 0$. We prove the existence and uniqueness of solutions of such equations as well as some regularity results. We also study stability of solutions with respect to the convergence of measures on the right-hand side of the equation. For this purpose, we introduce some type of functional convergence of smooth measures, which in fact is equivalent to the quasi-uniform convergence of associated potentials.

1. Introduction

Let E be a separable locally compact metric space, $(\mathcal{E}, D[\mathcal{E}])$ be a regular symmetric Dirichlet form on $L^2(E;m)$ and let μ be a nontrivial (i.e. $\mu(E) > 0$) positive Borel measure on E. In the present paper we study elliptic equations of the form

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http://dx.doi.org/10.1016/j.jfa.2016.10.029
0022-1236/

$$-Au = g(u) \cdot \mu, \quad u > 0, \tag{1.1}$$

where A is the operator associated with $(\mathcal{E}, D[\mathcal{E}])$ and $g : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+$ is a continuous function satisfying

$$c_1 \le g(u) \cdot u^\gamma \le c_2, \quad u > 0 \tag{1.2}$$

for some $c_1, c_2, \gamma > 0$. The model example of (1.1) is the Dirichlet problem

$$\begin{cases} -\Delta^{\alpha/2} u = u^{-\gamma} \cdot \mu, \quad u > 0, & \text{on } D, \\ u = 0 & \text{on } \mathbb{R}^d \setminus D, \end{cases} \tag{1.3}$$

where $\alpha \in (0, 2]$, $\gamma > 0$ and D is a bounded open subset of $\mathbb{R}^d$.

The paper consists of two parts. In the first part we address the problem of existence, uniqueness and regularity of solutions of (1.1). In the second part we study stability of solutions of (1.1) with respect to the convergence of measures on the right-hand side of the equation. The above problems were treated in [4] in case $A = \Delta$ and in [3] in case A is a uniformly elliptic divergence form operator. Some different but related problems are studied in [21] in case A is a Leray–Lions type operator. The main aim of the present paper is to generalize the results of [3,4] to equations with general (possibly nonlocal) operators corresponding to symmetric Dirichlet forms. We also refine some results proved in [3,4,21] for equations with local operators.

In the first part of the paper (Sections 3 and 4) we assume that μ belongs to the class $\mathcal{R}$ of smooth (with respect to capacity associated with $(\mathcal{E}, D[\mathcal{E}])$) positive Borel measures on E whose potential is m-a.e. finite (see Section 2 for details). It is known (see [16, Proposition 5.13]) that $\mathcal{M}_{0,b} \subset \mathcal{R}$, where $\mathcal{M}_{0,b}$ is the class of bounded smooth measures on E. In general, the inclusion is strict. For instance, in case of (1.3), $\mathcal{R}$ includes smooth Radon measures μ such that $\int_D \delta^{\alpha/2}(x)\, \mu(dx) < \infty$, where $\delta(x) = \text{dist}(x, \partial D)$ (see [14, Example 5.2]).

The first difficulty we encounter when considering equation (1.1) is to define properly a solution. Here we give a probabilistic definition of a solution of (1.1) via the Feynman–Kac formula. Namely, by a solution of (1.1) we mean a quasi-continuous function u on E such that $u > 0$ quasi-everywhere (q.e. for short) with respect to the capacity Cap naturally associated with $(\mathcal{E}, D[\mathcal{E}])$ and for q.e. $x \in E$,

$$u(x) = E_x \int_0^\zeta g(u)(X_t)\, dA_t^\mu.$$

Here $\{(X_t)_{t \ge 0}, (P_x)_{x \in E}\}$ is a Hunt process with life time ζ associated with the form $(\mathcal{E}, D[\mathcal{E}])$, E_x is the expectation with respect to P_x and A^μ is the positive continuous additive functional in the Revuz correspondence with μ.