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Iterative actions of normal operators ☆

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Let A be a normal operator in a Hilbert space $\mathcal{H}$, and let $\mathcal{G} \subset \mathcal{H}$ be a countable set of vectors. We investigate the relations between A, $\mathcal{G}$ and L that make the system of iterations $\{A^n g : g \in \mathcal{G},\ 0 \le n < L(g)\}$ complete, Bessel, a basis, or a frame for $\mathcal{H}$. The problem is motivated by the dynamical sampling problem and is connected to several topics in functional analysis, including, frame theory and spectral theory. It also has relations to topics in applied

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Sampling
Dynamics

harmonic analysis including, wavelet theory and time-frequency analysis.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space, $A \in B(\mathcal{H})$ a bounded normal operator and $\mathcal{G}$ a countable (finite or countably infinite) collection of vectors in $\mathcal{H}$. Let L be a function $L : \mathcal{G} \to \mathbb{N}^*$, where $\mathbb{N}^* = \{1, 2, \dots\} \cup \{+\infty\}$. We are interested in the structure of the set of iterations of the operator A when acting on the vectors in $\mathcal{G}$ and are limited by the function L. More precisely, we are interested in the following two questions:

(I) Under what conditions on A, $\mathcal{G}$ and L is the iterated system of vectors

$$\{A^n g : g \in \mathcal{G},\ 0 \le n < L(g)\}$$

complete, Bessel, a basis, or a frame for $\mathcal{H}$?

(II) If $\{A^n g : g \in \mathcal{G},\ 0 \le n < L(g)\}$ is complete, Bessel, a basis, or a frame for $\mathcal{H}$ for some system of vectors $\mathcal{G}$ and a function $L : \mathcal{G} \to \mathbb{N}^*$, what can be deduced about the operator A?

We study these and other related questions and we give answers in many important and general cases. In particular, we show that there is a direct relation between the spectral properties of a normal operator and the properties of the systems of vectors generated by its iterative actions on a set of vectors. We hope that the questions above and the approach we use can be interesting for research in both, frame theory and operator theory.

For the particular case when $L(g) = \infty$ for every $g \in \mathcal{G}$, we show that, if the system of iterations $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is complete and Bessel, then the spectral radius of A must be less than or equal to 1. Since A is normal, A must be a contraction in this case, i.e., $\|A\| \le 1$. Moreover, its unitary part must be absolutely continuous (with respect to the Lebesgue measure on the circle). The converse of this is also true: for every normal contraction with absolutely continuous unitary part, there exists a set $\mathcal{G}$ such that $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is a complete Bessel system. If $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is a frame then its unitary part must be 0 and the converse is also true: for every normal contraction with no unitary part there exists a set $\mathcal{G}$ such that $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is a (Parseval) frame.

The questions above, in their formulation have similarities with problems involving cyclical vectors in operator theory, and our analysis relies on the spectral theorem for normal operators with multiplicity [12]. There have been some attempts to generalize