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On the mean-field approximation of many-boson dynamics

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ABSTRACT

We show under general assumptions that the mean-field approximation for quantum many-boson systems is accurate. Our contribution unifies and improves most of the known results. The proof uses general properties of quantization in infinite dimensional spaces, phase-space analysis and measure transportation techniques.

1. Introduction

The mean-field theory provides a fair approximation of the dynamics and the ground state energies of many-body quantum systems. For nearly two decades, the subject has attracted a significant attention from the mathematical physics community and has become widely studied to this day. Mainly one can distinguish between boson and fermion

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systems with several physically motivated scaling regimes [20]. In particular, for bosonic systems there are at least three different regimes that depend on the range of the inter-atomic interaction and which yield the following mean-field equations:

(i) The Gross–Pitaevskii equation (see for instance [15,21,22,26,30]).
(ii) The NLS equation discussed for instance in [1,3,20,17].
(iii) The Hartree equation which is our main interest here.

The first contributions on the derivation of the Hartree dynamics were achieved in [25,28,43]. Several methods and results have since been elaborated, see for instance [11,12,19,20,24,31,42]. While the question of accuracy of the mean-field approximation is now well-understood for the most significant examples of quantum mechanics, it has no satisfactory general mathematical answer. In fact, most of the known works in the subject deal with a specific model or a specific choice of quantum states. Our aim here is to show that the mean-field approximation for bosonic systems is rather a general principle that depends very little on these above-mentioned specifications.

To enlighten the discussion, we recall the meaning of the convergence of many-body quantum dynamics towards the Hartree evolution in a concrete example and postpone the abstract framework to the previous section. Generally, the Hamiltonian of many-boson systems has the following form,

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V(x_i) + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} W(x_i - x_j)\,, \quad x_i, x_j \in \mathbb{R}^d\,, \tag{1}$$

where V is a real measurable function, W is a real potential, satisfying $W(-x) = W(x)$, $x \in \mathbb{R}^d$. It is in principle meaningful to include multi-particles interactions but to keep the presentation simple we avoid to do so (see [7,35,16]). Since we are dealing with bosons we assume that H_N is a self-adjoint operator on the symmetric tensor product space $L^2_s(\mathbb{R}^{dN})$. Recall that $L^2_s(\mathbb{R}^{dN})$ is the space of square integrable functions which are invariant with respect to any permutations of the coordinates $(x_1, \cdots, x_n) \in \mathbb{R}^{dN}$ with $x_i \in \mathbb{R}^d$ for $i = 1, \cdots, N$. Suppose that the many-body quantum system is prepared at an initial state $|\Psi^{(N)}\rangle\langle\Psi^{(N)}|$, such that $\Psi^{(N)} \in L^2_s(\mathbb{R}^{dN})$, then according to the Heisenberg equation the time-evolved state is given by,

$$\varrho_N(t) := |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}|\,.$$

The mean-field approximation provides the first asymptotics of physical measurements in the state $\varrho_N(t)$ when the number of particles N is sufficiently large. Precisely, the approximation deals with the following quantities,

$$\lim_{N\to\infty} \mathrm{Tr}[\varrho_N(t)\, B \otimes 1^{\otimes(N-k)}]\,, \tag{2}$$

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