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A Carleson type inequality for fully nonlinear elliptic equations with non-Lipschitz drift term

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ABSTRACT

This paper concerns the boundary behavior of solutions of certain fully nonlinear equations with a general drift term. We elaborate on the non-homogeneous generalized Harnack inequality proved by the second author in [26], to prove a generalized Carleson estimate. We also prove boundary Hölder continuity and a boundary Harnack type inequality.

1. Introduction

In this paper we study the boundary behavior of solutions of the following non-homogeneous, fully nonlinear equation

$$F(D^2u, Du, x) = 0\,. \tag{1.1}$$

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The operator F is assumed to be elliptic in the sense that there are $0 < \lambda \leq \Lambda$ such that

$$\lambda \mathrm{Tr}(Y) \leq F(X, p, x) - F(X + Y, p, x) \leq \Lambda \mathrm{Tr}(Y)\,, \quad \forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N \tag{1.2}$$

for every pair of symmetric matrices X, Y where Y is positive semidefinite. We assume that F has a drift term which satisfies the following growth condition

$$\big|F(0, p, x)\big| \leq \phi(|p|)\,, \quad \forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N \tag{1.3}$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous, increasing, and satisfies the structural conditions from [26] (see Section 2). Note that the function $F(\cdot, p, x)$ is 1-homogeneous while $F(0, \cdot, x)$ in general is not. In the case there is no drift term, i.e., $\phi = 0$, we say that the equation (1.1) is homogeneous.

The problem we are interested in is the so-called Carleson estimate [12]. The Carleson estimate can be stated for the Laplace equation in modern notation as follows. Let $\Omega \subset \mathbb{R}^N$ be a sufficiently regular bounded domain and $x_0 \in \partial\Omega$. If u is a non-negative harmonic function in $B(x_0, 4R) \cap \Omega$ which vanishes continuously on $\partial\Omega \cap B(x_0, 4R)$, then

$$\sup_{B(x_0, R/C) \cap \Omega} u \leq C u(A_R)\,, \tag{1.4}$$

where the constant C depends only on $\partial\Omega$ and N, and where $A_R \in B(x_0, R/C) \cap \Omega$ such that $d(A_R, \partial\Omega) > R/C^2$ (A_R is usually called a corkscrew point). For Ω to be sufficiently regular it is enough to assume that Ω is e.g., an NTA-domain, see [24]. The Carleson estimate is very important and useful when studying the boundary behavior and free boundary problems for linear elliptic equations [11,13,24,28], for p-Laplace type elliptic equations [5–7,32–35], for parabolic p-Laplace type equations [3,4], and for homogeneous fully nonlinear equations [19–21].

In this paper we deal with either Lipschitz or $C^{1,1}$ domains and assume that they are locally given by graphs in balls centered at the boundary with radius up to $R_0 > 0$ which unless otherwise stated satisfies $R_0 \leq 16$. For a given Lipschitz domain with Lipschitz constant l we denote $L = \max\{l, 2\}$. The main result of this paper is the sharp Carleson type estimate for non-negative solutions of (1.1). Due to the non-homogeneity of the equation it is easy to see that (1.4) cannot hold. Instead the Carleson estimate takes a similar form as the generalized interior Harnack inequality proved in [26] (see Theorem 2.1). Our main result reads as follows.

Theorem 1.1. *Assume that Ω is a Lipschitz domain such that $0 \in \partial\Omega$ and assume $u \in C(B_{4R} \cap \overline{\Omega})$, with $R \in (0, R_0/4]$, is a non-negative solution of (1.1). Let $A_R \in B_{R/2L} \cap \Omega$ be a point such that $d(A_R, \partial\Omega) > R/(4L^2)$, and assume that $u = 0$ on $\partial\Omega \cap B_{4R}$. There exists a constant $C > 1$ which is independent of u and of the radius R such that*