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On second order elliptic and parabolic equations of mixed type

Gong Chen [a,*,1], Mikhail Safonov [b]

[a] Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
[b] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

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ABSTRACT

It is known that solutions to second order uniformly elliptic and parabolic equations, either in divergence or nondivergence (general) form, are Hölder continuous and satisfy the interior Harnack inequality. We show that even in the one-dimensional case ($x \in \mathbb{R}^1$), these properties are not preserved for equations of mixed divergence–nondivergence structure: for elliptic equations.

$$D_i(a^1_{ij}D_ju) + a^2_{ij}D_{ij}u = 0,$$

and parabolic equations

$$p\partial_t u = D_i(a_{ij}D_ju),$$

where $p = p(t, x)$ is a bounded strictly positive function. The Hölder continuity and Harnack inequality are known if p does not depend either on t or on x. We essentially use homogenization techniques in our construction. Bibliography: 22 titles.

* Corresponding author.
E-mail addresses: gc@math.uchicago.edu (G. Chen), safonov@math.umn.edu (M. Safonov).

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1. Introduction

Leaving aside the elliptic and parabolic equations with "regular" coefficients, and also the cases of lower dimension, the Hölder regularity of solutions was first proved in 1957 by De Giorgi [3] for uniformly elliptic equations, and soon afterwards by Nash [19] for more general uniformly parabolic equations in the *divergence form*

$$Lu := -\partial_t u + D_i\big(a_{ij}D_j u\big) = 0, \tag{D}$$

where $\partial_t u := \partial u/\partial t$, $D_i := \partial/\partial x_i$ for $i = 1, 2, \ldots, n$, and the equation is understood in the integral sense, i.e. u is a *weak solution* of (D). Here and throughout the paper, we assume the summation convention over repeated indices i, j from 1 to n. The coefficients $a_{ij} = a_{ij}(t, x)$ are real Borel measurable functions satisfying the *uniform parabolicity condition*

$$a_{ij}\xi_i\xi_j \geq \nu\,|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n, \quad \text{and} \quad \sum_{i,j} a_{ij}^2 \leq \nu^{-2}, \tag{U}$$

with $\nu = \text{const} \in (0, 1]$. The elliptic equations $D_i\big(a_{ij}D_j u\big) = 0$ can be formally considered as a particular case of (D), in which a_{ij} and u do not depend on t, so that $\partial_t u = 0$.

The interior Harnack inequality was first proved in 1961 by Moser [16] for elliptic equation, and then in 1964 [17] for more general parabolic equations (D) (see Theorem 2.1 below).

The Harnack inequality together with Hölder regularity of solutions to uniformly parabolic equations in the *nondivergence form*

$$Lu := -\partial_t u + a_{ij}D_{ij}u = 0 \tag{ND}$$

was proved much later, at the end of 1970th, by Krylov and Safonov [11]. See [10,7,14,20] and references therein for further history, generalizations to equations with unbounded lower order terms, and various applications. The method in [11] is based on some variants of *growth lemmas*, which were originally introduced by Landis [13].

More recently, Ferretti and Safonov [4] tried to develop some "unifying" techniques which would equally applicable to equations in both (D) and (ND) forms. They found out the growth lemmas can serve as a common ground for the proof of the Harnack inequality and other related facts, though the methods of their proof are completely different in these two cases.

A natural question arises, whether or not the Harnack inequality, Theorem 2.1, and Hölder estimate, Corollary 2.3, hold true, with constants independent on the smoothness of the coefficients, for solutions of mixed type elliptic equations of "mixed" type

$$D_i(a_{ij}^1 D_j u) + a_{ij}^2 D_{ij}u = 0, \tag{1.1}$$