Journal of Functional Analysis 272 (2017) 3259–3280

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

On the Kuznetsov trace formula for $\mathrm{PGL}_2(\mathbb{C})$

Zhi Qi

Department of Mathematics, Rutgers University, Hill Center, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

A R T I C L E   I N F O

Article history:
Received 3 March 2016
Accepted 29 January 2017
Available online 3 February 2017
Communicated by P. Delorme

MSC:
11F72
11M36

Keywords:
The Kuznetsov trace formula
Spectral decomposition
Bessel functions

A B S T R A C T

In this note, using a representation theoretic method of Cogdell and Piatetski-Shapiro, we prove the Kuznetsov trace formula for an arbitrary discrete group Γ in $\mathrm{PGL}_2(\mathbb{C})$ that is cofinite but not cocompact. An essential ingredient is a kernel formula, recently proved by the author, on Bessel functions for $\mathrm{PGL}_2(\mathbb{C})$. This approach avoids the difficult analysis in the existing method due to Bruggeman and Motohashi.

1. Introduction

In the paper [17], Kuznetsov discovered his trace formula for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2 \cong \mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})/K$, where $\mathbb{H}^2$ denotes the hyperbolic upper half-plane and $K = \mathrm{SO}(2)/\{\pm 1\}$. There are two forms of his formula. The approach to the first formula is through a spectral decomposition formula for the inner product of *two* (spherical) Poincaré series. Then, using an inversion formula for the Bessel transform, Kuznetsov obtained another form of his trace formula (the version in [7] is more complete in the sense that holomorphic cusp forms occurring in the Petersson trace formula are also

E-mail address: zhi.qi@rutgers.edu.

http://dx.doi.org/10.1016/j.jfa.2017.01.011
0022-1236/

involved). On the geometric side, a weighted sum of Kloosterman sums arises from computing the Fourier coefficients of Poincaré series. The spectral side involves the Fourier coefficients of holomorphic and Maaß cusp forms and Eisenstein series along with the Bessel functions associated with their spectral parameters. The second form plays a primary role in the investigation of Kuznetsov on sums of Kloosterman sums in the direction of the Linnik–Selberg conjecture.[1]

Along the classical lines, the Kuznetsov trace formula has been studied and generalized by many authors (see, for example, [1,2,22,7,4]). Their ideas of generalizing the formula to the non-spherical case are essentially the same as Kuznetsov. It should however be noted that the pair of Poincaré series is chosen and spectrally decomposed in the space of a given K-type.

In the framework of representation theory, the second form of the Kuznetsov formula for an arbitrary Fuchsian group of the first kind $\Gamma \subset \mathrm{PGL}_2(\mathbb{R})$ was proved straightforwardly by Cogdell and Piatetski-Shapiro [6]. Their computations use the Whittaker and Kirillov models of irreducible unitary representations of $\mathrm{PGL}_2(\mathbb{R})$. They observe that the Bessel functions occurring in the Kuznetsov formula should be identified with the Bessel functions for irreducible unitary representations of $\mathrm{PGL}_2(\mathbb{R})$ given by [6, Theorem 4.1], in which the Bessel function $\mathcal{J}_\pi$ associated with a $\mathrm{PGL}_2(\mathbb{R})$-representation π satisfies the following kernel formula,

$$W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\right) = \int_{\mathbb{R}^\times} \mathcal{J}_\pi(ab) W\begin{pmatrix} b & \\ & 1 \end{pmatrix} d^\times b, \tag{1.1}$$

for all Whittaker functions W in the Whittaker model of π. Note that their idea of approaching Bessel functions for $\mathrm{GL}_2(\mathbb{R})$ using local functional equations for $\mathrm{GL}_2 \times \mathrm{GL}_1$-Rankin–Selberg zeta integrals over $\mathbb{R}$ (see [6, §8]) also occurs in [23].

In the book of Cogdell and Piatetski-Shapiro [6], the Kuznetsov trace formula is derived from computing the Whittaker functions (Fourier coefficients) of a (*single*) Poincaré series $P_f(g)$ in two different ways, first unfolding $P_f(g)$ to obtain a weighted sum of Kloosterman sums, and secondly spectrally expanding $P_f(g)$ in $L^2(\Gamma\backslash G)$ and then computing the Fourier coefficients of the spectral components in terms of basic representation theory of $\mathrm{PGL}_2(\mathbb{R})$. The Poincaré series in [6] arise from a very simple type of functions that are supported on the Bruhat open cell of $\mathrm{PGL}_2(\mathbb{R})$ and split in the Bruhat coordinates, which is surely not of a fixed K-type. In other words, [6] suggests that, instead of the Iwasawa coordinates, it would be more pleasant to study the Kuznetsov formula using the Bruhat coordinates. For this, [6] works with the full spectral theorem rather than a version, used by all the other authors, that is restricted to a given K-type.

In the direction of generalization to other groups, Miatello and Wallach [21] gave the spherical Kuznetsov trace formula for real semisimple groups of real rank one, which

[1] It is Selberg who introduced Poincaré series and realized the intimate connections between Kloosterman sums and the spectral theory of the Laplacian on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$ in [24].