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Lower estimates of heat kernels for non-local Dirichlet forms on metric measure spaces ☆

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ABSTRACT

We are concerned with heat kernel estimates for a non-local Dirichlet form on an Ahlfors regular metric measure space. We use an analytic approach to obtain the full *lower* stable-like estimate of the heat kernel from the *near diagonal lower* estimate. Combining with other known results, we obtain certain equivalent conditions for two-sided stable-like estimates of the heat kernel. The results can be simplified in certain cases, for example, when the Dirichlet form admits an effective resistance.

Contents

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1. Introduction

In recent years there has been a growing interest in heat kernel estimates for *non-local* Dirichlet forms in various settings, see, for example [4,7,9–13,23,27] and the references therein. Let (M, d, μ) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. In many cases of interest, if $(\mathcal{E}, \mathcal{F})$ is of jump type, the heat kernel $p_t(x, y)$ of the associated heat semigroup admits the following two-sided estimate:

$$p_t(x,y) \asymp c \min\left(t^{-\alpha/\beta}, \frac{t}{d(x,y)^{\alpha+\beta}}\right), \tag{1.1}$$

where α is the Hausdorff dimension of the underlying metric space, β is a positive parameter called the *index*, and the symbol $\asymp$ means that both $\leq$ and $\geq$ hold but with different values of a positive constant c. For instance, the heat kernel of the fractional Laplacian $(-\Delta)^{\beta/2}$ on $\mathbb{R}^n$ for any $\beta \in (0, 2)$ admits the estimate (1.1) with $\alpha = n$. Note that $(-\Delta)^{\beta/2}$ is the generator of a symmetric stable process of index β. Hence, we refer to (1.1) as a *stable-like* estimate.

An important problem in this area is to provide equivalent conditions for the stable-like estimate (1.1) in reasonable terms, in particular, in terms of the jump kernel of the Dirichlet form. In the case $\beta < 2$ this problem was solved by Chen and Kumagai in [12,13][1] by showing that, under the standing assumption of α-regularity of (M, d, μ), the heat kernel estimate (1.1) is equivalent to the following estimate of the jump kernel of the Dirichlet form: $J(x, y) \asymp C d(x, y)^{-(\alpha+\beta)}$. However, on fractal spaces the index β in (1.1) can actually be larger than 2. The present paper is a part of the project of the authors of creating tools for obtaining stable-like estimate (1.1) for arbitrary $\beta > 0$ including the case $\beta > 2$.

Let us mention for comparison that the theory for obtaining heat kernel bounds for *local* Dirichlet forms on fractal-like spaces has reached by now a certain maturity. It is known that typically the heat kernel of a diffusion on such spaces satisfies *sub-Gaussian* estimate

[1] Although the results of [12,13] require some additional assumptions on the metric space (M, d), this method can be enhanced to work with a general metric structure as it is done in [29].