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Blow-up analysis concerning singular Trudinger–Moser inequalities in dimension two

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ABSTRACT

In this paper, we derive a sharp version of the singular Trudinger–Moser inequality, which was originally established by Adimurthi and Sandeep [2]. Moreover, extremal functions for those singular Trudinger–Moser inequalities are also obtained. Our method is the blow-up analysis. Compared with our previous work ([32]), the essential difficulty caused by the presence of singularity is how to analyse the asymptotic behavior of certain maximizing sequence near the blow-up point. We overcome this difficulty by combining two different classification theorems of Chen and Li [6,7] to get the desired bubble.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $W_0^{1,2}(\Omega)$ be a completion of $C_0^\infty(\Omega)$ under the norm $\|u\|_{W_0^{1,2}(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{1/2}$. The Sobolev embedding theorem states that $W_0^{1,2}(\Omega)$ is embedded in $L^p(\Omega)$ for any $p > 1$, but not in $L^\infty(\Omega)$. However, as a limit case of the Sobolev embedding, the Trudinger–Moser inequality [37,22,21,26,20] says that

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$$\sup_{u\in W_0^{1,2}(\Omega),\, \|\nabla u\|_2\le 1} \int_\Omega e^{\gamma u^2}dx < +\infty, \quad \forall \gamma \le 4\pi; \tag{1}$$

Moreover, these integrals are still finite for all $\gamma > 4\pi$ and all $u \in W_0^{1,2}(\Omega)$, but the supremum is infinity. This inequality was generalized in many ways, one of which is as below. Let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplacian, namely

$$\lambda_1(\Omega) = \inf_{u\in W_0^{1,2}(\Omega),\, u\not\equiv 0} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}. \tag{2}$$

Here and throughout this paper, we denote the usual $L^p(\Omega)$-norm by $\|\cdot\|_p$ for any $p > 0$. It was proved by Adimurthi and O. Druet [1] that for any $\alpha < \lambda_1(\Omega)$,

$$\sup_{u\in W_0^{1,2}(\Omega),\, \|\nabla u\|_2\le 1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}dx < +\infty; \tag{3}$$

Moreover the supremum is infinity for any $\alpha \ge \lambda_1(\Omega)$. This result was extended by Y. Yang [28,29] to the cases of high dimension and compact Riemannian surface, by Lu–Yang [19] and J. Zhu [38] to the version of L^p-norm, by de Souza and J.M. do Ó [11,13] to the whole Euclidean space, and by Tintarev [25] to the following form

$$\sup_{u\in W_0^{1,2}(\Omega),\, \|u\|_{1,\alpha}\le 1} \int_\Omega e^{4\pi u^2}dx < +\infty, \quad \forall \alpha < \lambda_1(\Omega). \tag{4}$$

Here and throughout this paper, for any α and u satisfying $\sqrt{\alpha}\|u\|_2 \le \|\nabla u\|_2$, we denote

$$\|u\|_{1,\alpha} = \left(\int_\Omega |\nabla u|^2dx - \alpha\int_\Omega u^2dx\right)^{1/2}. \tag{5}$$

One can check that (4) is stronger that (3). In a recent work [32], we generalized the inequality (4) to the case that large eigenvalues are involved, as well as to the manifold case. Also, we obtained extremal functions for these kind of Trudinger–Moser inequalities. For pioneer works on extremal functions for Trudinger–Moser inequality, we refer the reader to L. Carleson and A. Chang [8], M. Struwe [23], M. Flucher [14], K. Lin [18], and Y. Li [16].

Now we describe another kind of generalization of (1), namely the singular Trudinger–Moser inequality. Based on a rearrangement argument, Adimurthi and K. Sandeep [2] were able to prove the following: Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and $0 \le \beta < 1$ be fixed. Then there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\, \|\nabla u\|_2\le 1} \int_\Omega \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}}dx < +\infty. \tag{6}$$