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The Szlenk index of injective tensor products and convex hulls

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ABSTRACT

Given any Banach space X and any weak*-compact subset K of X^*, we compute the Szlenk index of the weak*-closed, convex hull of K as a function of the Szlenk index of K. Also as an application, we compute the Szlenk index of any injective tensor product of two operators. In particular, we compute the Szlenk index of an injective tensor product $X\hat{\otimes}_\varepsilon Y$ in terms of $Sz(X)$ and $Sz(Y)$. As another application, we give a complete characterization of those ordinals which occur as the Szlenk index of a Banach space, as well as those ordinals which occur as the Bourgain ℓ_1 or c_0 index of a Banach space.

1. Introduction

Since Szlenk introduced his index [27] to prove the non-existence of a separable, reflexive Banach space which is universal for the class of separable, reflexive Banach spaces, the Szlenk index has become an important tool in Banach space theory. For a survey of these results, we refer the reader to [18]. One remarkable property of the Szlenk index is that it perfectly determines the isomorphism classes of separable $C(K)$ spaces, which can be seen by combining the results of Bessaga and Pełczyński [4], Milutin [20], and

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Samuel [25]. In the sequel, we shall use w^* in place of weak* to refer to the weak* topology on a dual Banach space. The Szlenk index is also an important tool regarding asymptotically uniformly smooth and w^*-UKK renormings [16], and ξ–w^*-UKK renormings of separable Banach spaces [19]. For these questions, the relationship between the Szlenk index of a given set and the Szlenk index of its w^*-closed, convex hull is important. The content of this work provides a sharp result regarding the relationship between these two indices.

Let us recall the definition of the Szlenk derivation and the Szlenk index. Given a Banach space X, a w^*-compact subset K of X^*, and $\varepsilon > 0$, we let $s_\varepsilon(K)$ consist of all those $x^* \in K$ such that for all w^*-neighborhoods V of x^*, $\text{diam}(V \cap K) > \varepsilon$. We define the transfinite derived sets by $s^0_\varepsilon(K) = K$, $s^{\xi+1}_\varepsilon(K) = s_\varepsilon(s^\xi_\varepsilon(K))$, and $s^\xi_\varepsilon(K) = \cap_{\zeta<\xi} s^\zeta_\varepsilon(K)$ when ξ is a limit ordinal. We let $Sz_\varepsilon(K) = \min\{\xi : s^\xi_\varepsilon(K) = \varnothing\}$ if this class of ordinals is non-empty, and we write $Sz_\varepsilon(K) = \infty$ otherwise. We agree to the convention that $\xi < \infty$ for any ordinal ξ. We let $Sz(K) = \sup_{\varepsilon>0} Sz_\varepsilon(K)$, with the convention that the supremum is ∞ if $Sz_\varepsilon(K) = \infty$ for some $\varepsilon > 0$. Last, we let $Sz(X) = Sz(B_{X^*})$ for a Banach space X. Given an operator $A : X \to Y$, we let $Sz(A) = Sz(A^* B_{Y^*})$.

We also recall the definition of the Cantor–Bendixson derivative of a subset of a topological space. If K is a topological space and $L \subset K$, we let L' consist of all members of L which are not isolated in L. We define the transfinite derived sets L^ξ for all ordinals ξ as above: $L^0 = L$, $L^{\xi+1} = (L^\xi)'$, and $L^\xi = \cap_{\zeta<\xi} L^\zeta$ when ξ is a limit ordinal. We let $i(L)$ denote the minimum ξ such that $L^\xi = \varnothing$ if such a ξ exists, and $i(L) = \infty$ otherwise. We recall that L is *scattered* if and only if $i(L) < \infty$.

Throughout, **Ord** denotes the class of ordinal numbers, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We denote the first infinite ordinal by ω and the first uncountable ordinal by ω_1. Recall that a gamma number is an ordinal which is greater than the sum of any two lesser ordinals. Of course, 0 is a gamma number. The non-zero gamma numbers are precisely the ordinals of the form ω^ξ for some ξ [21]. Given an ordinal ξ, we let $\Gamma(\xi)$ denote the minimum gamma number which is not less than ξ. Since $\omega^\xi \geqslant \xi$ for any ordinal ξ, this minimum exists. For completeness, we agree that $\Gamma(\infty) = \infty$. For convenience, we will also agree to call ∞ a gamma number.

The main tool of this work is the following.

Theorem 1.1. *Let X be any Banach space, $K \subset X^*$ w^*-compact, and let $L = \overline{co}^*(K)$. Then $Sz(L) = \Gamma(Sz(K))$.*

Using the geometric version of the Hahn–Banach theorem (or in some cases the Krein–Milman theorem), Theorem 1.1 will allow us to compute the Szlenk index of a given set L by the Szlenk index of a subset of L whose w^*-closed, convex hull is L and whose Szlenk index is easy to compute. Our first example of such a phenomenon is the following.

Theorem 1.2. *Let K be any compact, Hausdorff topological space. Then $Sz(C(K)) = \Gamma(i(K))$.*