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Asymptotic behavior of the growth-fragmentation equation with bounded fragmentation rate

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ABSTRACT

We are interested in the large time behavior of the solutions to the growth-fragmentation equation. We work in the space of integrable functions weighted with the principal dual eigenfunction of the growth-fragmentation operator. This space is the largest one in which we can expect convergence to the steady size distribution. Although this convergence is known to occur under fairly general conditions on the coefficients of the equation, we prove that it does not happen uniformly with respect to the initial data when the fragmentation rate in bounded. First we get the result for fragmentation kernels which do not form arbitrarily small fragments by taking advantage of the Dyson–Phillips series. Then we extend it to general kernels by using the notion of quasi-compactness and the fact that it is a topological invariant.

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0. Introduction

In this article, we study the asymptotic behavior of the *growth-fragmentation equation*

$$\begin{cases} \partial_t f(t,x) + \partial_x \left(\tau(x) f(t,x)\right) = \mathcal{F} f(t,x), & t,x>0, \\ (\tau f)(t,0) = 0, & t>0, \\ f(0,x) = f^{\mathrm{in}}(x), & x \geq 0. \end{cases} \tag{1}$$

This equation appears in the modeling of various physical or biological phenomena [19,25,2,11] as well as in telecommunication [5]. The unknown $f(t,x)$ represents the concentration at time t of some "particles" with "size" $x>0$, which can be for instance the volume of a cell, the length of a fibrillar polymer, or the window size in data transmission over the Internet. Each particle grows with a rate $\tau(x)$ and splits according to the fragmentation operator $\mathcal{F}$ which acts on a function $f(x)$ through

$$\mathcal{F} f(x) := \mathcal{F}_+ f(x) - B(x) f(x).$$

The positive part $\mathcal{F}_+$ is an integral operator given by

$$\mathcal{F}_+ f(x) := \int_0^1 B\Big(\frac{x}{z}\Big) f\Big(\frac{x}{z}\Big) \frac{\wp(\mathrm{d}z)}{z}. \tag{2}$$

When a particle of size x breaks with rate $B(x)$, it produces smaller particles of sizes zx with $0<z<1$ distributed with respect to the fragmentation kernel $\wp$. The fragmentation kernel $\wp$ is a finite positive measure on the open interval $(0,1)$ which satisfies

$$\int_0^1 z\, \wp(\mathrm{d}z) = 1. \tag{3}$$

This is a mass conservation condition since it ensures that if we sum the sizes of the offsprings we recover the size of the mother particle.

Classical examples of fragmentation kernels are the mitosis kernel $\wp = 2\delta_{1/2}$, the asymmetrical division kernels $\wp = \delta_\nu + \delta_{1-\nu}$ with $\nu \in (0,1/2)$, and the power law kernels $\wp(\mathrm{d}z) = (\nu+2) z^\nu \mathrm{d}z$ with $\nu > -2$. Notice that the power law kernels are physically relevant only for $\nu \leq 0$ (see *e.g.* discussion in Section 8.2.1 of [2]), which includes the uniform kernel $\wp \equiv 2$.

The long time behavior of the solutions is strongly related to the existence of (λ, G, ϕ) solution to the following Perron eigenvalue problem:

$$(\tau G)' + \lambda G = \mathcal{F} G, \qquad G \geq 0, \qquad \int_0^\infty G(x)\,\mathrm{d}x = 1 \tag{4}$$