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Analytic deformations of group commuting squares and complex Hadamard matrices

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ABSTRACT

Let G be a finite group and denote by $\mathfrak{C}_G$ the commuting square associated to G. The defect of the group G, given by the formula $d(G) = \sum_{g \in G} \frac{|G|}{order(g)}$, was introduced in [9] as an upper bound for the number of linearly independent directions in which $\mathfrak{C}_G$ can be continuously deformed in the class of commuting squares. In this paper we show that this bound is actually attained, by constructing $d(G)$ analytic families of commuting squares containing $\mathfrak{C}_G$.

In the case $G = \mathbb{Z}_n$, the defect $d(\mathbb{Z}_n)$ can be interpreted as the dimension of the enveloping tangent space of the real algebraic manifold of $n \times n$ complex Hadamard matrices, at the Fourier matrix F_n (in the sense of [14,1]). The dimension of the enveloping tangent space gives a natural upper bound on the number of continuous deformations of F_n by complex Hadamard matrices, of linearly independent directions of convergence. Our result shows that this bound is reached, which is rather surprising. In particular our construction yields new analytic families of complex Hadamard matrices stemming from F_n.

In the last section of the paper we use a compactness argument to prove non-equivalence (i.e. non-isomorphism as commuting

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squares) for dephased versions of the families of Hadamard matrices constructed throughout the paper.

1. Introduction

Commuting squares were introduced in [10], as invariants and construction data in Jones' theory of subfactors [3,4]. They encode the generalized symmetries of the subfactor, in a lot of situations being complete invariants [11,10]. In particular, any finite group G can be encoded in a group commuting square:

$$\mathfrak{C}_G = \begin{pmatrix} D & \subset & \mathrm{M_n}(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}I_n & \subset & \mathbb{C}[G] \end{pmatrix}$$

where $D \simeq l^\infty(G)$ is the algebra of $n \times n$ diagonal matrices, and $\mathbb{C}[G]$ denotes the group algebra of G. It can be shown that two group commuting squares are isomorphic if and only if the corresponding groups are isomorphic. The subfactor associated to $\mathfrak{C}_G$ by iterating Jones' basic construction is a cross product subfactor, hence of depth 2. Moreover, if G is abelian then $\mathfrak{C}_G$ is a spin model commuting square, and the associated subfactor is a Hadamard subfactor in the sense of [7].

In [6], the first author initiated a study of the deformations of a commuting square, in the class of commuting squares. It was shown that if a commuting square satisfies a certain *span condition*, then it is isolated among all non-isomorphic commuting squares. In the case of $\mathfrak{C}_G$, the span condition is $V = \mathrm{M_n}(\mathbb{C})$, where V is the subspace of $\mathrm{M_n}(\mathbb{C})$ given by:

$$V = \mathrm{span}\{du - ud : d \in D, u \in \mathbb{C}[G]\} + \mathbb{C}[G] + \mathbb{C}[G]' + D$$

When the span condition fails, the dimension $d'(G)$ of $V^\perp = \mathrm{M_n}(\mathbb{C}) \ominus \mathrm{V}$ can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by non-isomorphic commuting squares. In [9] we computed this dimension, which we called *the dephased defect of the group G*. We also studied the related quantity $d(G) = \dim_{\mathbb{C}}([D, \mathbb{C}[G]]^\perp)$, called *the undephased defect of G* (or just the defect of G), which can be interpreted as an upper bound for the number of independent directions in which $\mathfrak{C}_G$ can be deformed by (not necessarily non-isomorphic) commuting squares. The terminologies 'dephased defect' and 'undephased defect' are based on previous work of [5], [14] and [1], which we explain below.

The concept of defect for unitary matrices can be traced back to [5]. The terminology 'defect' was first explicitly introduced in [14]. The (dephased) defect of the Fourier matrix $F_n = \frac{1}{\sqrt{n}}(e^{i\frac{2\pi kl}{n}})_{1\le k,l\le n}$ was computed, and it was proved that it gives an upper bound