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Pointwise bounds and blow-up for systems of semilinear parabolic inequalities and nonlinear heat potential estimates

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A B S T R A C T

We study the behavior for t small and positive of $C^{2,1}$ nonnegative solutions $u(x,t)$ and $v(x,t)$ of the system

$$\begin{aligned} 0 \le u_t - \Delta u \le v^\lambda \\ 0 \le v_t - \Delta v \le u^\sigma \end{aligned} \qquad \text{in } \Omega \times (0,1),$$

where λ and σ are nonnegative constants and Ω is an open subset of $\mathbb{R}^n$, $n \ge 1$. We provide optimal conditions on λ and σ such that solutions of this system satisfy pointwise bounds in compact subsets of Ω as $t \to 0^+$. Our approach relies on new pointwise bounds for nonlinear heat potentials which are the parabolic analog of similar bounds for nonlinear Riesz potentials.

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1. Introduction

In this paper we study the behavior for t small and positive of $C^{2,1}$ nonnegative solutions $u(x,t)$ and $v(x,t)$ of the system

$$\begin{aligned} 0 \le u_t - \Delta u \le v^\lambda \\ 0 \le v_t - \Delta v \le u^\sigma \end{aligned} \quad \text{in } \Omega \times (0,1), \tag{1.1}$$

where λ and σ are nonnegative constants and Ω is an open subset of $\mathbb{R}^n$, $n \ge 1$. More precisely, we consider the following question.

Question 1. For which nonnegative constants λ and σ do there exist continuous functions $h_1, h_2 : (0,1) \to (0,\infty)$ such that for all compact subsets K of Ω and for all $C^{2,1}$ nonnegative solutions $u(x,t)$ and $v(x,t)$ of the system (1.1) we have

$$\max_{x\in K} u(x,t) = O(h_1(t)) \quad \text{as } t \to 0^+ \tag{1.2}$$

$$\max_{x\in K} v(x,t) = O(h_2(t)) \quad \text{as } t \to 0^+ \tag{1.3}$$

and what are the optimal such h_1 and h_2 when they exist?

We call a function h_1 (resp. h_2) with the above properties a pointwise bound in compact subsets for u (resp. v) as $t \to 0^+$.

Remark 1.1. Let

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty) \\ 0 & \text{for } (x,t) \in \mathbb{R}^n \times (-\infty, 0] \end{cases} \tag{1.4}$$

be the heat kernel. Since $\Phi_t - \Delta\Phi = 0$ in $\mathbb{R}^n \times (0,\infty)$, the functions $u_0 = v_0 = \Phi$ are always $C^{2,1}$ nonnegative solutions of (1.1). Hence, since $\Phi(0,t) = \frac{1}{(4\pi t)^{n/2}}$, any pointwise bound as $t \to 0^+$ in compact subsets of Ω for nonnegative solutions of (1.1) must be at least as large as $t^{-n/2}$ and whenever $t^{-n/2}$ is such a bound for u (resp. v) it is necessarily optimal. In this case we say that u (resp. v) is *heat bounded* in compact subsets of Ω as $t \to 0^+$.

We shall see that whenever a pointwise bound as $t \to 0^+$ in compact subsets of Ω for nonnegative solutions of (1.1) exists, then u or v (or both) are heat bounded as $t \to 0^+$.

The literature on scalar and systems of parabolic *equations* is quite vast. A good source for this material is the book [12]. However much less attention has been paid to systems of parabolic inequalities. We mention results in [3,4,11] on the nonexistence