Journal of Functional Analysis 272 (2017) 1625–1660

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Magnetic Schrödinger operators on periodic discrete graphs

Evgeny Korotyaev [a], Natalia Saburova [b,*]

[a] *Saint-Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034, Russia*

[b] *Northern (Arctic) Federal University, Severnaya Dvina Emb. 17, Arkhangelsk, 163002, Russia*

A R T I C L E I N F O

Article history:
Received 27 December 2015
Accepted 13 December 2016
Available online 23 December 2016
Communicated by C. De Lellis

Keywords:
Spectral bands
Flat bands
Discrete magnetic Schrödinger operator
Periodic graph

A B S T R A C T

We consider magnetic Schrödinger operators with periodic magnetic and electric potentials on periodic discrete graphs. The spectrum of the operators consists of an absolutely continuous part (a union of a finite number of non-degenerate bands) plus a finite number of flat bands, i.e., eigenvalues of infinite multiplicity. We estimate the Lebesgue measure of the spectrum in terms of the Betti numbers and show that these estimates become identities for specific graphs. We estimate a variation of the spectrum of the Schrödinger operators under a perturbation by a magnetic field in terms of magnetic fluxes. The proof is based on Floquet theory and a precise representation of fiber magnetic Schrödinger operators constructed in the paper.

Contents

1. Introduction 1626

* Corresponding author.

E-mail addresses: korotyaev@gmail.com, e.korotyaev@spbu.ru (E. Korotyaev), n.saburova@gmail.com, n.saburova@narfu.ru (N. Saburova).

http://dx.doi.org/10.1016/j.jfa.2016.12.015

2. Main results . . . 1633
3. Direct integrals for magnetic Schrödinger operators . . . 1639
4. Proof of the main results . . . 1644
5. Properties of fiber operators and an example . . . 1649
6. Generalized magnetic Schrödinger operators . . . 1653
Acknowledgments . . . 1659
References . . . 1659

1. Introduction

1.1. Introduction

We discuss spectral properties of Schrödinger operators with periodic magnetic and electric potentials on $\mathbb{Z}^d$-periodic discrete graphs, $d \geqslant 2$, and in particular, magnetic Laplacians. The spectrum of these operators consists of an absolutely continuous part (a union of a finite number of non-degenerate bands) plus a finite number of flat bands, i.e., eigenvalues of infinite multiplicity. There are a lot of results about such problems, see e.g., [12,14,19,20,32] and the references therein.

A discrete analogue of the magnetic Laplacian on $\mathbb{R}^2$ was originally introduced by Harper [12]. This discrete magnetic Laplacian Δ_α acts on functions $f \in \ell^2(\mathbb{Z}^2)$, $n = (n_1, n_2) \in \mathbb{Z}^2$, and is given by:

$$\begin{aligned}(\Delta_\alpha f)(n) = 4f(n) &- e^{-iB\frac{n_2}{2}} f(n+e_1) - e^{iB\frac{n_2}{2}} f(n-e_1)\\ &- e^{-iB\frac{n_1}{2}} f(n+e_2) - e^{iB\frac{n_1}{2}} f(n-e_2),\end{aligned} \tag{1.1}$$

where $e_1 = (1,0)$, $e_2 = (0,1) \in \mathbb{R}^2$. The operator Δ_α describes the behavior of an electron moving on the square lattice $\mathbb{Z}^2$ exposed to a uniform magnetic field in the so-called tight-binding model [3]. The magnetic field $\mathcal{B} = B(0,0,1) \in \mathbb{R}^3$ with amplitude $B \in \mathbb{R}$ is perpendicular to the lattice. The corresponding vector potential α of the uniform magnetic field $\mathcal{B}$ is given by

$$\alpha(\mathbf{e}) = \begin{cases} -\frac{Bn_2}{2}, & \text{if } \mathbf{e} = (n, n+e_1) \\ -\frac{Bn_1}{2}, & \text{if } \mathbf{e} = (n, n+e_2) \end{cases}. \tag{1.2}$$

The value B is the magnetic flux through the unit cell of the lattice for the magnetic field $\mathcal{B}$. Note that the discrete magnetic Laplacian Δ_α is reduced to the Harper operator in the discrete Hilbert space $\ell^2(\mathbb{Z})$. Seemingly it is a very simple operator but, compared with the magnetic Laplacian on $\mathbb{R}^2$, its spectrum is very sensitive to the parameter B (see [2,6,9,20] and the references therein):

1) if $\frac{B}{2\pi}$ is a rational number, then the spectrum $\sigma(\Delta_\alpha)$ of the magnetic Laplacian Δ_α has a *band structure*, i.e., $\sigma(\Delta_\alpha)$ consists of a finite number of closed intervals;