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On the error term of a lattice counting problem

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We improve the error terms of some estimates related to counting lattices from recent work of L. Fukshansky, P. Guerzhoy and F. Luca (2017). This improvement is based on some analytic techniques, in particular on bounds of exponential sums coupled with the use of Vaaler polynomials.

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1. Introduction

1.1. Background

For integer $T \geq 1$, we let

$$\mathcal{F}(T) = \{a/b \ : \ (a,b) \in \mathbb{Z}^2, \ 0 \leq a < b \leq T, \ \gcd(a,b) = 1\}$$

be the set of Farey fractions. We also define

$$\mathcal{I}(T) = \mathcal{F}(T) \cap [0, 1/2] \,.$$

Now, following [6], we consider the quantity

$$C(T) = \sum_{a/b \in \mathcal{I}(T)} \#\mathcal{C}_{a,b}(T),$$

where

$$\mathcal{C}_{a,b}(T) = \mathcal{F}(T) \cap [1 - a^2/b^2, 1].$$

The quantity $C(T)$ appears naturally in some counting problems for two-dimensional lattices. More precisely, every similarity class of *planar lattices* can be parametrised by a point $\tau = x_0 + iy_0$ in

$$\mathcal{R} = \{\tau = x_0 + iy_0 \ : \quad 0 \leq x_0 \leq 1/2, \quad y_0 \geq 0, \quad |\tau| \geq 1\} \subseteq \mathbb{C},$$

where one identifies $\tau \in \mathcal{R}$ with the lattice

$$\Lambda_\tau = \begin{pmatrix} 1 & x_0 \\ 0 & y_0 \end{pmatrix} \mathbb{Z}^2.$$

Further, similarity classes of *arithmetic* planar lattices correspond to Λ_τ, where

$$\tau = a/b + i\sqrt{c/d}$$

for integers a, b, c, d such that

$$\gcd(a,b) = \gcd(c,d) = 1, \quad 0 \leq a \leq b/2, \quad d > 0, \quad c/d \geq 1 - a^2/b^2.$$

The class is *semistable* if furthermore $c \leq d$. With these conventions, the quantity $C(T)$ counts the number of similarity classes of semi-stable arithmetic planar lattices of height at most T, that if for which $\max\{a, b, c, d\} \leq T$.