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Another generalization of a theorem of Baker and Davenport ☆

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ABSTRACT

Dujella and Pethő, generalizing a result of Baker and Davenport, proved that the set $\{1, 3\}$ cannot be extended to a Diophantine quintuple. As a consequence of our main result, we show that the Diophantine pair $\{1, b\}$ is regular if $b - 1$ is a prime power.

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Pell equation
Linear forms in logarithms

1. Introduction

A set of m distinct positive integers $\{a_1, \ldots, a_m\}$ is called a *Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square. Diophantus studied sets of positive rational numbers with the same property, particularly he found the set of four positive rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. But the first Diophantine quadruple was found by Fermat observing that the set $\{1, 3, 8, 120\}$ is a Diophantine quadruple. Moreover, Baker and Davenport, in their classical paper [1], proved that the set $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple.

In 1997, Dujella [3] obtained that the Diophantine triples of the form $\{k-1, k+1, 4k\}$, for $k \geq 2$, cannot be extended to a Diophantine quintuple. The Baker–Davenport's result corresponds to $k = 2$. In 1998, Dujella and Pethő [6] proved that the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple. In 2008, Fujita [7] obtained a more general result by showing that the Diophantine pair $\{k-1, k+1\}$ cannot be extended to a Diophantine quintuple for any integer $k \geq 2$. In 2004, Dujella [5] proved that there are only finitely many Diophantine quintuples. A folklore conjecture states that there does not exist a Diophantine quintuple. This conjecture was proved by the first, third authors and V. Ziegler [9]. Let

$$d_+ = d = a + b + c + 2abc + 2\sqrt{(ab+1)(ac+1)(bc+1)}.$$

A stronger version of this conjecture is the following

Conjecture. *If $\{a, b, c, d\}$ is a Diophantine quadruple and $d > \max\{a, b, c\}$, then $d = d_+$.*

We introduce the concept of *regular quadruple*. A Diophantine quadruple $\{a, b, c, d\}$ is regular if and only if $(a + b - c - d)^2 = 4(ab + 1)(cd + 1)$. Therefore, the quadruples in the above conjecture are regular.

The (solved) folklore conjecture is weaker than the above conjecture. For a fixed Diophantine triple $\{a, b, c\}$, we view the fourth element $x > \max\{a, b, c\}$ as a solution such that $\{a, b, c, x\}$ is a Diophantine quadruple. A trivial solution is $x = d_+$. The latest result on the folklore conjecture (see [9]) lets us know that there are no two such solutions $x = x_1$ $(= d_+)$ and $x = x_2$ corresponding to the same Diophantine triple $\{a, b, c\}$ with an additional condition that $x_1 x_2 + 1 = \square$. But, in general when we remove the relation between x_1 and x_2, we don't know whether $x = d_+$ is the unique solution. We do not even know whether there are infinitely many irregular Diophantine quadruples or not. Namely, this conjecture is still open.

The aim of this paper is to consider the extensibility of the Diophantine pair $\{1, b\}$ and to give a new generalization of the mentioned theorems by Baker and Davenport [1]