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Menon-type identities derived from actions of subgroups of general linear groups

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ABSTRACT

In this article, we consider the actions of subgroups of the general linear group $GL_r(\mathbb{Z}_n)$ on $\mathbb{Z}_n^r$, including groups of upper triangular matrices in $GL_r(\mathbb{Z}_n)$, unipotent groups, Heisenberg groups and extended Heisenberg groups. By applying the Cauchy–Frobenius–Burnside lemma, we obtain several generalizations of the well known Menon's identity.

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1. Introduction

In [3], P.K. Menon proved the following elegant identity. For every positive integer n, we have

$$\sum_{a\in U(\mathbb{Z}_n)} \gcd(a-1,n) = \varphi(n)\tau(n),$$

where $U(\mathbb{Z}_n) = \{a \in \mathbb{Z}_n \,|\, \gcd(a,n) = 1\}$ is the group of units of the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, φ is the Euler's totient function and $\tau(n)$ is the number of positive divisors of n.

Many authors obtained various generalizations of Menon's identity (e.g., see [1,2,4,5,7,6,10,11,14]). A useful technique to prove Menon-type identities is the so-called Cauchy–Frobenius–Burnside lemma (see [9] or [8, Chapter 3]), which states that

Cauchy–Frobenius–Burnside lemma: Let G be a finite group acting on a finite set X. Let $G\backslash X$ be the set of orbits and $X^g = \{x \in X | gx = x\}$ be the set of fixed elements in X by g, where $g \in G$. Then

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

That is, the number of distinct orbits is the average number of fixed points by the elements of the group.

For example, the Menon's identity is derived from considering the action of $U(\mathbb{Z}_n)$ on $\mathbb{Z}_n$.

In [12], B. Sury computed both sides of the Cauchy–Frobenius–Burnside lemma for the action of the multiplicative matrix group

$$G = \left\{ \left. \begin{pmatrix} a & b_1 & b_2 & \dots & b_r \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \right| a \in U(\mathbb{Z}_n), b_i \in \mathbb{Z}_n \text{ for all } i > 1 \right\}$$

on the set $X = \mathbb{Z}_n^{r+1}$ as matrix multiplication on the left of column vectors. Then, Sury obtained the following identity:

$$\sum_{\substack{a\in U(\mathbb{Z}_n)\\ b_1,\dots,b_r\in\mathbb{Z}_n}} \gcd(a-1,b_1,\dots,b_r,n) = \varphi(n)\sigma_r(n),$$

where $\sigma_r(n) = \sum_{d|n} d^r$ is the sum of r-th powers of positive divisors of n. He also considered the action of the general linear group $GL_r(\mathbb{Z}_n)$ on $\mathbb{Z}_n^r$ and derived some results (see [12, Theorem 2]).