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Jacobi forms and differential operators: Odd weights

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ABSTRACT

We show that it is possible to remove two differential operators from the standard collection of m of them used to embed the space of Jacobi forms of *odd* weight k and index m into several pieces of elliptic modular forms. This complements the previous work of one of the authors in the case of even weights.

1. Introduction

Let $J_{k,m}(N)$ denote the space of Jacobi forms of weight k and index m for the Jacobi group $\Gamma_0(N) \ltimes \mathbb{Z}^2$ of level N. It is well known that certain differential operators D_ν ($0 \le \nu \le m$ for k even and $1 \le \nu \le m-1$ for k odd), first systematically studied in the monograph by Eichler–Zagier (see [4]), map $J_{k,m}(N)$ injectively into a direct

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sum of finitely many spaces of elliptic modular forms. See section 2 for a description of these objects. This result has many applications, estimating the dimension of $J_{k,m}(N)$ precisely is one of them. Moreover information about the vanishing of the kernel of D_0 when $k = 2, m = 1$ and N is square-free, has a bearing on the Hashimoto's conjecture on theta series, see [1,2].

When the index $m = 1$, the question about $\ker D_0$, which is nothing but the restriction map from $J_{k,m}(N)$ to $M_k(N)$, the space of elliptic modular forms of weight k on $\Gamma_0(N)$; defined by $\phi(\tau, z) \mapsto \phi(\tau, 0)$, translates into the possibility of removing the differential operator D_2 while preserving injectivity. This question is also interesting in its own right, and has received some attention in the recent past, see the works [1–3,7], and the introduction there. We only note here that first results along this line of investigation seems to be by J. Kramer, who gave an explicit description of $\ker D_0$ when $m = 1$ and level N a prime, in terms of the vanishing order of cusp forms in a certain subspace of $S_4(N)$ (this is related to the so-called Weierstrass subspaces of $S_k(N)$, see [2]).

An unpublished question of S. Böcherer, inspired by the case $m = 1$ as discussed above, asks for information about this phenomenon when the index in bigger than 1. Let us recall that below.

Question. For $0 \le \nu \le m$ and $m \ge 1$, the map $i_\nu(k, m, N)$ defined by

$$D_0 \oplus \dots \widehat{D_\nu} \dots \oplus D_{2m} \colon J_{k,m}(N) \xrightarrow{i_\nu(k,m,N)} M_k(N) \oplus \dots \widehat{M_{k+\nu}(N)} \dots \oplus M_{k+2m}(N)$$

is **injective**; where the $\widehat{\ }$ signifies that the corresponding term has to be omitted.

Note that it is classical (see also section 2.2) that the above map without any term omitted is an injection, so the above question asks for something stronger. This was answered relatively satisfactorily in [3] when k was even, namely that one can, under certain conditions on m, k, remove the last operator D_{2m}.

In this paper, we take up the case k odd and henceforth assume this condition. As expected, there are some subtleties in this case. First of all, it is clear that only differential operators with odd indices matter, and moreover by [4, p. 37] it is known that the last operator D_{2m-1} can be removed, since an odd Jacobi form $\phi(\tau, z)$ cannot have a more than $(2m-3)$-fold zero at $z = 0$. This shows that the above **Question** in our context can be phrased as:

Question′. For $1 \le \nu \le m-1$, $m \ge 3$ with k odd and keeping the above notation, the map

$$J_{k,m}(N) \xrightarrow{i_{2\nu-1}(k,m,N)} M_{k+1}(N) \oplus \dots \widehat{M_{k+2\nu-1}(N)} \dots \oplus M_{k+2m-3}(N)$$

is **injective**.