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On a product of certain primes

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ABSTRACT

We study the properties of the product, which runs over the primes,

$$\mathfrak{p}_n = \prod_{s_p(n) \geq p} p \quad (n \geq 1),$$

where $s_p(n)$ denotes the sum of the base-p digits of n. One important property is the fact that $\mathfrak{p}_n$ equals the denominator of the Bernoulli polynomial $B_n(x) - B_n$, where we provide a short p-adic proof. Moreover, we consider the decomposition $\mathfrak{p}_n = \mathfrak{p}_n^- \cdot \mathfrak{p}_n^+$, where $\mathfrak{p}_n^+$ contains only those primes $p > \sqrt{n}$. Let $\omega(\cdot)$ denote the number of prime divisors. We show that $\omega(\mathfrak{p}_n^+) < \sqrt{n}$, while we raise the explicit conjecture that

$$\omega(\mathfrak{p}_n^+) \sim \kappa \frac{\sqrt{n}}{\log n} \quad \text{as } n \to \infty$$

with a certain constant $\kappa > 1$, supported by several computations.

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1. Introduction

Let $\mathbb{P}$ be the set of primes. Throughout this paper, p denotes a prime, and n denotes a nonnegative integer. The function $s_p(n)$ gives the sum of the base-p digits of n. Let $\mathcal{P}(n)$ denote the greatest prime factor of $n \geq 2$, otherwise $\mathcal{P}(n) = 1$. An empty product is defined to be 1.

We study the product of certain primes,

$$\mathfrak{p}_n := \prod_{\substack{p \in \mathbb{P} \\ s_p(n) \geq p}} p \quad (n \geq 1), \tag{1}$$

which is restricted by the condition $s_p(n) \geq p$ on each prime factor p. Since $s_p(n) = n$ in case $p > n$, the product (1) is always finite.

The values $\mathfrak{p}_n$ are of basic interest, as we will see in Section 2, since they are intimately connected with the denominators of the Bernoulli polynomials and related polynomials.

Theorem 2 below supplies sharper bounds on the prime factors of $\mathfrak{p}_n$. For the next theorem, giving properties of divisibility, we need to define the squarefree kernel of an integer as follows:

$$\operatorname{rad}(n) := \prod_{p \mid n} p, \quad \operatorname{rad}^*(n) := \begin{cases} 1, & \text{if } n \text{ is prime}, \\ \operatorname{rad}(n), & \text{else} \end{cases} \quad (n \geq 1).$$

Theorem 1. *The sequence $(\mathfrak{p}_n)_{n \geq 1}$ obeys the following divisibility properties:*

(a) *Any prime p occurs infinitely often:*

$$n \equiv -1 \pmod{p} \quad (n > p) \quad \Longrightarrow \quad p \mid \mathfrak{p}_n.$$

(b) *Arbitrarily large intervals of consecutive members exist such that*

$$p \mid \mathfrak{p}_n \quad \Longrightarrow \quad p \mid \mathfrak{p}_{np^r+b} \quad (0 \leq b < p^r,\ r \geq 1).$$

(c) *Arbitrarily many prime factors occur, in particular:*

$$\operatorname{rad}^*(n+1) \mid \mathfrak{p}_n.$$

Theorem 2 *(Kellner and Sondow [3]). If $n \geq 1$, then*

$$\mathfrak{p}_n = \prod_{\substack{p \leq \frac{n+1}{\lambda_n} \\ s_p(n) \geq p}} p,$$

where

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