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ON THE DISTRIBUTION OF POLYNOMIALS WITH BOUNDED HEIGHT

CSANÁD BERTÓK*, LAJOS HAJDU** AND ATTILA PETHŐ***

Dedicated to Professor Andrzej Schinzel on the occasion of his 80th birthday.

ABSTRACT. We provide an asymptotic expression for the probability that a randomly chosen polynomial with given degree, having integral coefficients bounded by some B, has a prescribed signature. We also give certain related formulas and numerical results along this line. Our theorems are closely related to earlier results of Akiyama and Pethő, and also yield extensions of recent results of Dubickas and Sha.

1. INTRODUCTION

Let d be a positive integer, $B \geq 1$ a real number. Denote by $\mathcal{H}_d(B)$ the set of $(d+1)$-dimensional vectors $(p_0, \ldots, p_d)$ satisfying $|p_i| \leq B$ $(0 \leq i \leq d), p_d \neq 0$. In the case $B = 1$ we write simply $\mathcal{H}_d$ instead of $\mathcal{H}_d(1)$.

Given a polynomial $P \in \mathbb{R}[X]$, the non-real roots of P appear in complex conjugate pairs. Thus $d = r + 2s$, where r denotes the number of real roots and s the number of non-real pairs of roots of P. As we shall work with arbitrary but fixed d and then r is uniquely determined by s, we call s the signature of P. Identifying the vector $(p_0, \ldots, p_d) \in \mathbb{R}^{d+1}$ and the polynomial $p_d X^d + p_{d-1} X^{d-1} + \cdots + p_0$ the set $\mathcal{H}_d(B)$ splits naturally into $\lfloor d/2 \rfloor + 1$ disjoint subsets according to the signature. In the sequel $\mathcal{H}_d(s, B)$ denotes the subset of $\mathcal{H}_d(B)$ whose elements have signature s. If $B = 1$, in place of $\mathcal{H}_d(s, 1)$ we shall simply write $\mathcal{H}_d(s)$. Plainly, $\mathcal{H}_d(s, B)$ is a bounded set in $\mathbb{R}^{d+1}$ for any $B > 0$, and we will prove that it is Lebesgue measurable. For the Lebesgue measure (which we shall often simply call volume) of $A \subset \mathbb{R}^n$ we write $\lambda_n(A)$ or $\lambda(A)$, if the dimension n is obvious.

Following Dubickas and Sha [4] denote by $\mathcal{D}_d^*(s, B)$[1] the set of polynomials $f(X) = p_d X^d + p_{d-1} X^{d-1} + \cdots + p_0 \in \mathbb{Z}[X]$ with $p_d \neq 0$, $|p_i| \leq B$ $(i =$

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[1] In fact Dubickas and Sha [4] called (r, s) the signature of P and used the notation $\mathcal{D}_d^*(r, s, B)$ instead of $\mathcal{D}_d^*(s, B)$. As we frequently cite the papers of Akiyama and Pethő [1] and [2], where only s was used for the signature and sets of polynomials were denoted according to this convention, we follow their notation.