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Rational approximations of the exponential function at rational points

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We give explicit and asymptotic lower bounds for the quantity $|e^{s/t} - M/N|$ by studying a generalized continued fraction expansion of $e^{s/t}$. In cases $|s| \geq 3$ we improve existing results by extracting a large common factor from the numerators and the denominators of the convergents of that generalized continued fraction.

1. Introduction

We will present both explicit and asymptotic irrationality measure results i.e. lower bounds for the quantity $|e^{s/t} - M/N|$ as a function of positive integer N, where s/t is an arbitrary non-zero rational number.

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As usual, the lower bounds are achieved by constructing sequences of high quality rational approximations to $e^{s/t}$. Our approximations are based on the convergents of the generalized continued fraction expansion

$$e^z = 1 + \frac{2z}{2-z+} \frac{z^2}{6} \frac{z^2}{+10} \frac{z^2}{+14+\ldots} = 1 + \frac{2z}{2-z+\mathrm{K}_{n=1}^{\infty} \frac{z^2}{4n+2}}, \tag{1}$$

which are in fact the diagonal Padé approximants of the exponential function [13,7,8]. Special attention will be paid for divisibility properties of the numerators and the denominators of the convergents of

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{z^2}{4n+2}$$

evaluated at $z = s/t$. There appears an unexpected and fairly big common factor which is related to the prime number decomposition of s. Thereby when $|s| \geq 3$, we improve the existing results considerably; the comparison will be done in the next section.

Let $s \in \mathbb{Z} \setminus \{0\}$, $t \in \mathbb{Z}_{\geq 1}$ and $\gcd(s,t) = 1$. Denote the inverse of the function $y(z) = z \log z$, $z \geq 1/e$, as $z(y)$. We define $z_0(y) = y$ and $z_n(y) = y/\log z_{n-1}(y)$, $n \in \mathbb{Z}_{\geq 1}$. In the following we denote

$$\alpha = \prod_{\substack{p \in \mathbb{P} \\ p|s}} p^{1/(p-1)}, \qquad \beta = \sum_{\substack{p \in \mathbb{P}\setminus\{2\} \\ p|s}} 1, \qquad \gamma = \prod_{\substack{p \in \mathbb{P}\setminus\{2\} \\ p|s}} p,$$

$$\sigma = \frac{4t\alpha}{es^2}, \qquad \rho = \begin{cases} \frac{7}{3}, & \sigma \geq 1, \\ 5 - 2\log\sigma, & \sigma < 1, \end{cases} \qquad \zeta(N) = \frac{z(\sigma \log N)}{\sigma} + \beta,$$

$$Z(N) = \left(\frac{s^2}{\alpha^2} (\zeta(N) + 1) \right)^{\beta} \times \left(\frac{8\gamma^2 |s| (4t\zeta(N) + 6t + s^2) \zeta(N)^{\beta}}{\alpha} + \frac{\gcd(2,s)\alpha^2\gamma}{N^2 s^2 |e^{s/t} - 1|} \right),$$

$$\eta = \max\left\{ \frac{\sqrt{e}|e^{s/t} - 1|\gamma}{\sqrt{2}\sigma^{\beta}} - \frac{1}{2}, \frac{e}{\sigma} + \beta \right\}, \qquad \varepsilon(N) = \frac{\log\log\log N}{\log\log N}.$$

Theorem 1.1. *Let $s \in \mathbb{Z} \setminus \{0\}$, $t \in \mathbb{Z}_{\geq 1}$ and $\gcd(s,t) = 1$. Then*

$$1 < \left| e^{s/t} - \frac{M}{N} \right| Z(N)\, N^{2 + 2\log(|s|/\alpha)\, z(\sigma \log N)/(\sigma \log N)}$$

for all $M \in \mathbb{Z} \setminus \{0\}, N \in \mathbb{Z}_{\geq N_1}$ with

$$\log N_1 = \max\left\{ (\eta - \beta)\log(\sigma(\eta - \beta)), \log\left(\frac{\gcd(2,s)}{|4s + 2(s-2t)(e^{s/t} - 1)|} \right) \right\}. \tag{2}$$