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SUMS OF TWO RATIONAL CUBES WITH MANY PRIME FACTORS

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Abstract. In this paper, we show that for any given integer $k \geq 2$, there are infinitely many cube-free integers n having exactly k prime divisors such that n is a sum of two rational cubes. This is a cubic analogue of the work of Tian [Ti], which proves that there are infinitely many congruent numbers having exactly k prime divisors for any given integer $k \geq 1$.

1. Introduction and results

Let n be a cube-free integer and $E_n : x^3+y^3 = n$ the elliptic curve defined over $\mathbb{Q}$. Let $L_{E_n}(s)$ be the Hasse-Weil L-function of E_n and $w_n \in \{1, -1\}$ its root number. Then $L_{E_n}(s)$ satisfies the functional equation

$$N^{s/2}(2\pi)^{-s}\Gamma(s)L_{E_n}(s) = w_n N^{(2-s)/2}(2\pi)^{-(2-s)}\Gamma(2-s)L_{E_n}(2-s),$$

where N is the conductor of E_n whose divisors are 3 and primes $p \mid n$. The *analytic rank* of E_n is the order of vanishing at the central point $s = 1$ of $L_{E_n}(s)$. The functional equation implies that $w_n = 1$ if and only if the analytic rank of E_n is even. The Birch and Swinnerton-Dyer(BSD) conjecture states that the rank of the Mordell-Weil group $E_n(\mathbb{Q})$ is equal to the analytic rank of E_n. So the BSD conjecture implies that if $w_n = -1$, then n is a sum of two rational cubes.

The root number w_n can be computed by the following way, due to Birch and Stephens [BS],

$$w_n = \prod_{p \text{ prime}} w_n(p),$$

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