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On rational approximations to two irrational numbers

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Let α and β be two irrational real numbers satisfying $\alpha \pm \beta \notin \mathbb{Z}$. We prove several inequalities between $\min_{k\in\{1,\dots,n\}} \|k\alpha\|$ and $\min_{k\in S} \|k\beta\|$, where S is a set of positive integers, e.g., $S = \{n\}$, $S = \{1, \dots, n-1\}$ or $S = \{1, \dots, n\}$ and $\|x\|$ stands for the distance between $x \in \mathbb{R}$ and the nearest integer. We also give some constructions of α and β which show that the result of Kan and Moshchevitin (asserting that the difference between $\min_{k\in\{1,\dots,n\}} \|k\alpha\|$ and $\min_{k\in\{1,\dots,n\}} \|k\beta\|$ changes its sign infinitely often) and its variations are best possible. Some of the results are given in terms of the sequence $d(n) = d_{\alpha,\beta}(n)$ defined as the difference between reciprocals of these two quantities. In particular, we prove that the sequence $d(n)$ is unbounded for any irrational α, β satisfying $\alpha \pm \beta \notin \mathbb{Z}$.

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1. Introduction

Given a real number ξ, let

$$\psi_\xi(n) := \min_{k \in \{1,2,\dots,n\}} \|k\xi\|$$

be the irrationality measure function of ξ, where $\|x\|$ stands for the distance between $x \in \mathbb{R}$ and the nearest integer. In [1], Kan and Moshchevitin proved that for any two irrational real numbers α, β satisfying $\alpha \pm \beta \notin \mathbb{Z}$ the difference

$$\psi_\alpha(n) - \psi_\beta(n)$$

is positive for infinitely many $n \in \mathbb{N}$ and also negative for infinitely many $n \in \mathbb{N}$. Although recently a metric result of this type for some matrices is given in [4], in the above mentioned paper [1], by some classical results of Khintchine (see Theorems 1 and 2 in [1] or the original paper of Kchintchine [2]), it was shown that the possibilities to extend this result to dimensions greater than 1 are very limited.

In this paper we first give a short proof of the next result (equivalent to that in [1]) and then show that this result and its variations are in some sense best possible.

Theorem 1. *Let α and β be two irrational real numbers satisfying $\alpha \pm \beta \notin \mathbb{Z}$. Then,*

$$\psi_\alpha(n) > \|n\beta\| \tag{1}$$

for infinitely many $n \in \mathbb{N}$.

Observe that (1) implies

$$\psi_\alpha(n) > \|n\beta\| \geqslant \psi_\beta(n)$$

for infinitely many $n \in \mathbb{N}$. Also, since the condition $\alpha \pm \beta \notin \mathbb{Z}$ is equivalent to $\beta \pm \alpha \notin \mathbb{Z}$, by reversing the roles of α and β, we derive that $\psi_\beta(n) > \|n\alpha\| \geqslant \psi_\alpha(n)$ for infinitely many $n \in \mathbb{N}$. In particular, this implies the main result of [1].

Although in the proof of Theorem 1 we use the language of combinatorics on words, the proof is essentially the same as in [1]. The same setting is also used in the proof of Theorem 5 below. However, the full advantage of this approach shows up in the proof of Theorem 2 which is much more complicated.

We remark that under condition $\alpha \pm \beta \in \mathbb{Z}$ we have $\|n\alpha\| = \|n\beta\|$ for each $n \in \mathbb{N}$, and hence $\psi_\alpha(n) = \psi_\beta(n)$ for each $n \in \mathbb{N}$.

Since $\psi_\alpha(n)$ and $\psi_\beta(n)$ both tend to zero as $n \to \infty$ (and so their difference tends to 0 as $n \to \infty$, but $\psi_\alpha(n) \neq 0$ and $\psi_\beta(n) \neq 0$ when $\alpha, \beta \notin \mathbb{Q}$), it is of interest to investigate not only the sequence of differences of two irrationality measures $\psi_\alpha(n) - \psi_\beta(n)$, $n = 1, 2, 3, \dots,$ but also the sequence of differences of their reciprocals