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The determinant of a double covering of the projective space of even dimension and the discriminant of the branch locus

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The determinant of the Galois action on the ℓ-adic cohomology of the middle degree of a proper smooth variety of even dimension defines a quadratic character of the absolute Galois group of the base field. In this article, we show that for a branched double covering of the projective space of even dimension, the character is computed via the square root of the discriminant of the defining polynomial of the double covering.

As a corollary, we deduce that the parity of a Galois permutation of the exceptional divisors on a Del Pezzo surface of degree 2 can be computed by the discriminant.

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Let k be a field, $\bar{k}$ an algebraic closure of k and k_s the separable closure of k contained in $\bar{k}$. Let $\Gamma_k = \mathrm{Gal}(k_s/k) = \mathrm{Aut}_k(\bar{k})$.

Let X be a proper smooth variety of even dimension n over k. If ℓ is a prime number invertible in k, the ℓ-adic cohomology $V = H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ defines an orthogonal representation of the absolute Galois group Γ_k. The determinant

$$\det V : \Gamma_k \to \{\pm 1\} \subset \mathbb{Q}_\ell^\times$$

is independent of the choice of ℓ (Corollary 2.2).

First we assume that char $k \neq 2$. Let f be a homogeneous polynomial of $n+1$ variables of even degree d with coefficients in k. Let X be the branched double covering of the projective space of dimension n defined by the equation $y^2 = f$. Then, the variety X is smooth if and only if the divided discriminant $\mathrm{disc}_d(f)$ (Definition 1.3) is not equal to zero. We prove the following theorem (Theorem 2.3).

Theorem 0.1. *If the double covering X is smooth, then the quadratic character $\det V$ is defined by the square root of $(-1)^{\frac{dn}{4}} \mathrm{disc}_d(f)$.*

[In other words, the kernel of $\det V : \Gamma_k \to \{\pm 1\}$ is the subgroup of Γ_k corresponding to the field extension $k\big(\sqrt{(-1)^{\frac{dn}{4}} \cdot \mathrm{disc}_d(f)}\,\big)/k$.]

We follow the method given by T. Saito in [6] where he computes $\det V$ for a smooth hypersurface of even dimension via the square root of the discriminant of a defining polynomial of the hypersurface. The proof of the Theorem 0.1 consists of two parts. First we follow the method given in [6], using a standard argument on universal family. Second, we then determine the $\det V$ by a specialization argument.

Next we assume that char $k = 2$. In this case, we consider more general polynomials $y^2 + ay = b$ defining branched double coverings where a and b are homogeneous polynomials of degree $\frac{d}{2}$ and d over k. Let

$$A = \sum_{|I| = \frac{d}{2}} R_I T^I, \quad B = \sum_{|J| = d} S_J T^J$$

be the universal polynomials of degree $\frac{d}{2}$ and d. Then, the discriminant $\mathrm{disc}_d(A^2 + 4B)$ is a polynomial of $(R_I)_{|I| = \frac{d}{2}}$ and $(S_J)_{|J| = d}$ of coefficients in $\mathbb{Z}$. The greatest common divisor of the coefficients of $\mathrm{disc}_d(A^2 + 4B)$ is $4^{d \cdot s(n,d)}$, where

$$s(n,d) = \frac{(n+1)(d-1)^n - a(n,d)}{d}, \quad a(n,d) = \frac{(d-1)^{n+1} - (-1)^{n+1}}{d}$$

(Lemma 3.3). Further, there exist polynomials C and D satisfying

$$4^{-d \cdot s(n,d)} (-1)^{\frac{dn}{4}} \mathrm{disc}_d(A^2 + 4B) = C^2 + 4D$$

(Theorem 3.6). We prove the following theorem (Theorem 3.7).