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Jack Klys

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REFLECTION PRINCIPLES FOR CLASS GROUPS

JACK KLYS

Abstract. We present several new examples of reflection principles which apply to both class groups of number fields and picard groups of of curves over $\mathbb{P}^1/\mathbb{F}_p$. This proves a conjecture of Lemmermeyer [3] about equality of 2-rank in subfields of A_4, up to a constant not depending on the discriminant in the number field case, and exactly in the function field case. More generally we prove similar relations for subfields of a Galois extension with group G for the cases when G is S_3, S_4, A_4, D_{2l} and $\mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$. The method of proof uses sheaf cohomology on 1-dimensional schemes, which reduces to Galois module computations.

Introduction

In this paper we look at the problem of relating the size of the l-torsion in the class groups of two distinct number fields. We let $\mathrm{rk}_l Cl\,(K) = \dim_{\mathbb{F}_l} Cl\,(K)\,[l]$.

Describing the size of the l-torsion of the class group of a number field is in general a hard problem. There are special cases where one can say something about $\mathrm{rk}_l Cl\,(K)$. For example it is easy to show for any quadratic field K that $\mathrm{rk}_2 Cl\,(K) = r-1$, where r is the number of primes ramified in K. One source of theorems describing l-torsion in class groups is Iwasawa theory, which gives formulas for $\mathrm{rk}_l Cl\,(K)$ for K lying in some tower of number fields, in terms of certain invariants depending on the base field. A very strong asymptotic conjecture on class group order is the following due to Zhang [8]:

Conjecture. *For any number field K let $n = [K : \mathbb{Q}]$ and let $\epsilon > 0$. Then*

$$|Cl\,(K)\,[l]| \ll_{\epsilon,n,l} D_K^{\epsilon}.$$

That is the l-torsion in number fields of a fixed degree grows slower than any power of their discriminant. Some work has been done in this direction by Ellenberg and Venkatesh [2] by combining reflection principles with analytic techniques.

A slightly different kind of problem is that of relating l-rank in class groups of two different number fields. Such statements are often called reflection principles. There are many such statements known and we will give a very brief overview of some of these. For a more exhaustive exposition we refer the reader to [3].

The following is referred to as the Scholz reflection theorem [7]:

Proposition 1. *Let $D > 1$ be square-free. Let $K = \mathbb{Q}(\sqrt{-3D})$ and $F = \mathbb{Q}(\sqrt{D})$. Then $\mathrm{rk}_l Cl(F) \leq \mathrm{rk}_l Cl(K) \leq \mathrm{rk}_l Cl(F) + 1$.*

There is the following special case of Leopoldt's theorem, originally due to Hecke [7]: