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An inverse problem about minimal zero-sum sequences over finite cyclic groups

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ABSTRACT

The article characterizes the minimal zero-sum sequences over the cyclic group C_n with lengths between $\lfloor n/3 \rfloor + 3$ and $\lfloor n/2 \rfloor + 1$, for $n \geq 10$. This is a step beyond established results about minimal zero-sum sequences over C_n of lengths at least $\lfloor n/2 \rfloor + 2$. The range of the obtained characterization is optimal.

Among the possible approaches we choose one with a strong emphasis on unsplittable sequences—intriguing objects generalizing the longest minimal zero-sum sequences over an abelian group. The unsplittable sequences over C_n with lengths in $[\lfloor n/3 \rfloor + 3, \lfloor n/2 \rfloor + 1]$ prove capable of capturing the essence of our setting and deserve an explicit description.

1. Introduction

A nonempty sequence over an additively written finite abelian group is a *minimal zero-sum sequence* if its sum is the zero element of the group and all of its proper nonempty subsequences have nonzero sums. There are satisfactory descriptions of the

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sufficiently long minimal zero-sum sequences over the cyclic group C_n, $n \geq 3$, the ones with lengths at least $\lfloor n/2 \rfloor + 2$ (see [9,13]). The obtained characterization theorems found a variety of applications.

We aim to extend these results to shorter sequences over C_n. Preliminary observations suggest that the minimal zero-sum sequences with lengths in $[\lfloor n/3 \rfloor + 3, \lfloor n/2 \rfloor + 1]$ are a natural choice.

Problems of this flavor are known as *inverse zero-sum problems*. They ask for an exhaustive description of sequences with prescribed properties, usually extremal ones. The highlight of this recently developed area is the inverse Davenport problem for finite abelian groups of rank 2, that is, the characterization of the longest minimal zero-sum sequences over the groups $C_m \oplus C_n$ ($m|n$, $m>1$). The path to its solution is long and complex. The formidable combined effort of Geroldinger, Gao, Grynkiewicz and Schmid over the span of several multipage articles [3,4,11] prepared the ground for Reiher's decisive final contribution in his dissertation [8]. Inverse zero-sum problems are particularly relevant to the theory of non-unique factorizations. Details can be found in the monograph of Geroldinger and Halter-Koch [7] and Geroldinger's survey [6].

Perhaps the simplest minimal zero-sum sequences over a finite abelian group G are the sequences α with terms of the form $t = x_t a$, where $a \in G$ is nonzero with order $\mathrm{ord}(a)$, the x_t are positive integers and $\sum_{t\in\alpha} x_t = \mathrm{ord}(a)$. We call $\{a\}$ a 1-*element basis* for α. All "long" minimal zero-sum sequences over C_n are of this form. More precisely, one of the equivalent ways to state the main result in [9] and [13] is: *Every minimal zero-sum sequence over C_n, $n \geq 3$, with length at least $\lfloor n/2 \rfloor + 2$ has a 1-element basis.*

There are sequences of any length with a 1-element basis. So such sequences are an inevitable part, albeit trivial, of the answer to the analogous problem for the range $[\lfloor n/3 \rfloor + 3, \lfloor n/2 \rfloor + 1]$. One approach to the nontrivial part stems from the definition of a 1-element basis. The very fact that α is a minimal zero-sum sequence is expressed by the equality $\sum_{t\in\alpha} x_t = \mathrm{ord}(a)$. We want a similar attribute in the characterization of the "shorter long" minimal zero-sum sequences. Here is an abridged version of a suitable definition. Its precise form is given in Section 4 (Definition 4.2).

Let the elements $u \neq 0$ and v of a finite abelian group G and $r \in \mathbb{Z}$ satisfy $2v = ru$, $0 < r \leq \mathrm{ord}(u)$ and $v \notin \{u, 2u, \ldots, ru\}$. We call the ordered pair (u, v) a 2-*element basis* for a sequence α over G if the terms of α can be represented in the form $t = x_t v + y_t u$ with $x_t \in \{0, 1\}$, $y_t \in \mathbb{Z}$, so that the equality $\sum_{t\in\alpha}(rx_t + 2y_t) = |\langle u, v\rangle|$ holds true.

The quantity $rx_t + 2y_t$ assumes the rôle of x_t in the definition of a 1-element basis; $\langle u, v\rangle$ is the subgroup generated by u and v. Here we omit the details of Definition 4.2. Let us note only that the sum $\sum_{t\in\alpha}(rx_t + 2y_t)$ can have at most one negative summand. A sequence with a 2-element basis is a minimal zero-sum sequence, although this is not immediately visible (Proposition 4.3).

Now the nontrivial part of our general characterization result (Theorem 7.9) can be stated as follows: