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On the binary additive divisor problem in mean

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We study a mean value of the classical additive divisor problem, that is

$$\sum_{f\sim F}\sum_{n\sim N}\left|\sum_{l\sim L} d(n+l)d(n+l+f) - \text{main term}\right|^2,$$

with quantities $N \geq 1$, $1 \leq F \ll N^{1-\varepsilon}$ and $1 \leq L \leq N$. The main term we are interested in here is the one by Motohashi [27], but we also give an upper bound for the case where the main term is that of Atkinson [1]. Furthermore, we point out that the proof yields an analogous upper bound for a shifted convolution sum over Fourier coefficients of a fixed holomorphic cusp form in mean.

1. Introduction

A widely studied occurrence of the classical divisor function $d(n) = \sum_{d|n} 1$ is in the additive divisor problem, in which one investigates the asymptotic behavior of the sum

$$D(x; m) = \sum_{n\leq x} d(n)d(n+m) \tag{1}$$

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as x tends to infinity and m is a given positive integer. Ingham was the first to find an asymptotic formula for the above sum in 1927, and in 1931 Estermann brought the Kloosterman sums into discussion, improving the earlier result. The importance of uniformity in the shift m was noted ten years later by Atkinson [1], who further used his findings in studying the fourth power mean of the Riemann zeta-function on the critical line. In 1982 Deshouillers and Iwaniec applied Kuznetsov's trace formulas into the problem, obtaining a notable improvement to the upper bound of the error term. Finally, especially crucial for this paper, Motohashi [27] worked out an explicit spectral decomposition for a weighted convolution sum, separating a main term and obtaining an upper bound for the error term uniformly in the shift m. We mention shortly papers by Heath-Brown, Iwaniec [11] and Ivić–Motohashi [9,10], who obtained mean value results for the error term, and [23,25,32] for other related results. For a more thorough discussion of the history and the references of the above mentioned classical papers we refer to Motohashi's comprehensive paper [27].

In recent years the analogy between the divisor function and the Fourier coefficients of holomorphic and non-holomorphic cusp forms has appeared repeatedly in the literature. The behavior of the analogous sums to (1),

$$A(x;m) = \sum_{n\le x} a(n)\overline{a(n+m)} \tag{2}$$

involving the Fourier coefficients of a holomorphic cusp form and

$$T(x;m) = \sum_{n\le x} t(n)t(n+m) \tag{3}$$

involving the Hecke eigenvalues corresponding to Fourier coefficients of a Maass form, has been studied intensively. In 1965 Selberg studied the meromorphity of a certain Dirichlet series related to the Fourier coefficients of holomorphic cusp forms, while Good in 1981 estimated the second moment of a modular L-function on the critical line using the spectral decomposition of shifted convolution sums coming from the Fourier coefficients of a holomorphic cusp form. Jutila [14,15] derived explicit asymptotic formulae for all three cases (1), (2) and (3) in a unified way, using the respective generating Dirichlet series. For a discussion of the substantial history including e.g. the works by Blomer, Duke, Friedlander, Harcos, Iwaniec, Jutila, Michel, Motohashi and Sarnak as well as a general spectral decomposition for the weighted shifted convolution sums we refer to the paper [2] by Blomer and Harcos. Along with these references we mention also [3,6,7,16,20]. Together the three sums above go under the name of the shifted convolution problem.

As a prologue for this paper, in Lemma 3 of his paper [16] Jutila studies the sum (2) in mean, and his argument can easily be extended to prove

$$\sum_{0\le f\le F}\sum_{1\le n\le N}\left|\sum_{1\le l\le L} a(n+l)\overline{a(n+f+l)}\right|^2 \ll (N+F)^k N^k L \tag{4}$$