Journal of Number Theory 178 (2017) 19–30

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Moments on quadratic binomial products

Wenchang Chu

Dipartimento di Matematica e Fisica 'Ennio De Giorgi', Università del Salento, Lecce–Arnesano PO Box 193, Lecce 73100, Italy

A R T I C L E I N F O

Article history:
Received 24 June 2016
Received in revised form 10 October 2016
Accepted 7 February 2017
Available online 31 March 2017
Communicated by D. Goss

MSC:
primary 11B65
secondary 05A10

Keywords:
Binomial coefficient
Catalan triangle
Chu–Vandermonde convolution
Telescoping method
Partial binomial sum

A B S T R A C T

We prove a general transformation theorem that expresses the moments on quadratic products of binomial coefficients as linear sums of their four initial values. Sixteen summation formulae are presented explicitly as examples. They contain, as special cases, the previous results due to Chen–Chu (2009) and Miana et al. (2007, 2008, 2010).

1. Introduction and motivation

In 1976, Shapiro [7] introduced Catalan triangles with the entries given by

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k} \quad \text{where} \quad k,\, n \in \mathbb{N} \quad \text{with} \quad k \le n. \tag{1}$$

E-mail address: chu.wenchang@unisalento.it.

http://dx.doi.org/10.1016/j.jnt.2017.02.005

Recently, Gutiérrez et al. [3] found the following two summation formulae

$$\sum_{k=1}^{n} kB_{n,k}^2 = \binom{n+1}{2} C_n C_{n-1}, \tag{2}$$

$$\sum_{k=1}^{n} k^2 B_{n,k}^2 = (3n-2)C_{2n-2}. \tag{3}$$

The more general problem of evaluating the binomial moments

$$\Theta_\gamma(n) = \sum_{k=1}^{n} k^\gamma \binom{2n}{n-k}^2 \tag{4}$$

was resolved by Chen–Chu [1]. The following counterparts of the above binomial moments

$$\Phi_\gamma(n) = \sum_{k=1}^{n+1} (2k-1)^\gamma \binom{2n+1}{n+k}^2 \tag{5}$$

are examined by Miana–Romero [6]. Similar binomial sums have been evaluated and extended subsequently by Guo–Zeng [2], Slavik [8], Sun–Ma [9] and Kilic–Prodinger [4].

Unifying both binomial moments displayed in (4) and (5), the purpose of the present paper is to investigate the following moments on quadratic product of binomial coefficients and evaluate them in closed forms:

$$\Theta_\gamma^\delta(m,n) := \sum_{k\geq\delta} \left(k-\tfrac{\delta}{2}\right)^\gamma \binom{2m+\delta}{m+k}\binom{2n+\delta}{n+k} \tag{6}$$

where $\delta = 0, 1$ and γ, m, n are three natural numbers.

As preliminaries, four summation formulae for $\Theta_\varepsilon^\delta$ with ε, $\delta = 0, 1$ will be established in the next section. They will be employed in Section 3 to prove a general transformation theorem that expresses Θ_γ^δ for $\gamma \in \mathbb{N}_0$ in terms of $\Theta_\varepsilon^\delta$ with ε, $\delta = 0, 1$. Sixteen summation formulae are explicitly presented as examples. They contain, as special cases, the previous results due to Chen–Chu [1] and Miana et al. [3,6].

2. Evaluation of four binomial sums

By means of the telescoping method and the Chu–Vandermonde convolution formula on binomial coefficients, this section will prove four summation formulae for $\Theta_\varepsilon^\delta$ with ε, $\delta = 0, 1$. Among them, we have even the fortune to evaluate the partial sums corresponding to $\delta = 1$ in closed forms. They will be utilized, in the next section to examine the binomial moments Θ_γ^δ for $\gamma \in \mathbb{N}_0$ in general.