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A weighted divisor problem [☆]

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ABSTRACT

We study a weighted divisor function

$$\sum_{mn\le x}{}' \cos(2\pi m\theta_1)\sin(2\pi n\theta_2),$$

where θ_i $(0 < \theta_i < 1)$ is a rational number. By connecting it with the divisor problem with congruence conditions, we establish an upper bound, mean-value, mean-square and some power-moments.

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1. Introduction and main results

1.1. Introduction

Let $d(n) = \sum_{n=n_1 n_2} 1$ denote the divisor function, and $D(x) = \sum'_{n \le x} d(n) = \sum'_{n_1 n_2 \le x} 1$ be the summatory function, where the prime $'$ on the summation sign indicates that if x is an integer, then only $\frac{1}{2}d(x)$ or $\frac{1}{2}$ for $n_1 n_2 = x$ is counted. In 1849, Dirichlet first proved that

$$D(x) = x \log x + (2\gamma - 1)x + O(\sqrt{x}), \forall x \ge 1,$$

where γ is the Euler constant.

Let

$$\Delta(x) = D(x) - x \log x - (2\gamma - 1)x - \frac{1}{4}$$

be the error term in the asymptotic formula for $D(x)$. Dirichlet's divisor problem consists of determining the smallest α, for which $\Delta(x) \ll x^{\alpha+\varepsilon}$ holds for any $\varepsilon > 0$. Clearly, Dirichlet's result above implies that $\alpha \le \frac{1}{2}$. Throughout the past more than 160 years, there have been many improvements on this estimate. The best estimate to-date has been given by Huxley [4,5], and reads

$$\Delta(x) \ll x^{\frac{131}{416}} \log^{\frac{26947}{8320}} x. \tag{1.1}$$

It is widely conjectured that $\alpha = \frac{1}{4}$ is admissible and the best possible.

Since $\Delta(x)$ exhibits considerable fluctuations, one natural way to study the upper bounds is to consider the moments.

Voronoi's work [17] in 1904 showed that

$$\int_1^X \Delta(x) dx = \frac{X}{4} + O(X^{\frac{3}{4}}).$$

Later, in 1922 Cramér [2] proved the mean square formula

$$\int_1^X \Delta(x)^2 dx = cX^{\frac{3}{2}} + O(X^{\frac{5}{4}+\varepsilon}), \quad \forall\, \varepsilon > 0,$$

where c is a positive constant. In 1983, Ivic [6] used the method of large values to prove that

$$\int_1^X |\Delta(x)|^A dx \ll X^{1+\frac{A}{4}+\varepsilon}, \quad \forall\, \varepsilon > 0 \tag{1.2}$$