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Trigonometric series and special values of L-functions

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Inspired by representations of the class number of imaginary quadratic fields, in this paper, we give explicit evaluations of trigonometric series having generalized harmonic numbers as coefficients in terms of odd values of the Riemann zeta function and special values of L-functions subject to the parity obstruction. The coefficients that arise in these evaluations are shown to belong to certain cyclotomic extensions. Furthermore, using best polynomial approximation of smooth functions under uniform convergence due to Jackson and their log-sine integrals, we provide approximations of real numbers by combinations of special values of L-functions corresponding to the Legendre symbol. Our method for obtaining these results rests on a careful study of generating functions on the unit circle involving generalized harmonic numbers and the Legendre symbol, thereby relating them to values of polylogarithms and then finally extracting Fourier series of special functions that can be expressed in terms of Clausen functions.

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1. Introduction

Generalized harmonic numbers of order $m \geq 1$ are defined as the sum

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}$$

for $n \geq 1$. The case $m = 1$ gives the classical harmonic numbers denoted by h_n. When $m \geq 2$, $H_n^{(m)}$ are the partial sums whose limit is the special value $\zeta(m)$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$ is the Riemann zeta function. Consider a series of the form

$$H(s, z) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{1}{k^z}$$

for complex s, z. Apostol and Vu [7] showed how to extend $H(s, z)$ to the complex plane as a meromorphic function. They have also discovered the reflection relation

$$H(s, z) + H(z, s) = \zeta(s)\zeta(z) + \zeta(s + z)$$

which holds for all complex s, z when both sides are finite. It turns out that such relations can be obtained from the more general identity

$$\sum_{m=1}^{\infty} a_m \sum_{k=1}^{m} b_k + \sum_{m=1}^{\infty} b_m \sum_{k=1}^{m} a_k = \sum_{m=1}^{\infty} a_m \sum_{m=1}^{\infty} b_m + \sum_{m=1}^{\infty} a_m b_m$$

which holds for any complex sequences $\{a_m\}$ and $\{b_m\}$ for which all these series are convergent. Special cases of this reflection were anticipated by Ramanujan [18] and by Sitaramachandra Rao and Sarma [20,21] (see also p. 252 of [8]). Moreover, values of the form $H(s, -j)$ for an integer $j \geq 0$ are expressed in terms of the Riemann zeta function (see [7]). Since then many different variations and extensions of such series were studied in the literature. These include the works of Sitaramachandra Rao [19], Mezö [17] and Boyadzhiev and Dil [10]. Given any character χ, the L-function associated to χ is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $\Re(s) > 1$. The parity obstruction related to the special values of $L(s, \chi)$ says that if $m \geq 2$ and χ are of opposite parity, then $L(m, \chi)$ can not be evaluated in terms of