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Ramanujan partition congruence modulo 13

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Partitions and powers of 13

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Abstract

In 1919, Ramanujan gave the identities

$$\sum_{n\geq 0} p(5n+4)q^n = 5\prod_{n\geq 1}\frac{(1-q^{5n})^5}{(1-q^n)^6}$$

and

$$\sum_{n\geq 0} p(7n+5)q^n = 7\prod_{n\geq 1}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n\geq 1}\frac{(1-q^{7n})^7}{(1-q^n)^8}$$

and in 1939, H.S. Zuckerman gave similar identities for $\sum_{n\geq 0} p(25n+24)q^n$, $\sum_{n\geq 0} p(49n+47)q^n$ and $\sum_{n\geq 0} p(13n+6)q^n$.

From Zuckerman's paper, it would seem that this last identity is an isolated curiosity, but that is not the case.

Just as the first four mentioned identities are well known to be the earliest instances of infinite families of such identities for powers of 5 and 7, the fifth identity is likewise the first of an infinite family of such identities for powers of 13. We will establish this fact and give the second identity in the infinite family.

1. Introduction

Throughout the paper, it should be understood that $E(q)$ is Euler's product,

$$E(q) = \prod_{n\geq 1}(1-q^n).$$

In 1919, Ramanujan [4] essentially proved that

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{E(q^5)^5}{E(q)^6} \tag{1}$$

and stated without proof that

$$\sum_{n\geq 0} p(7n+5)q^n = 7\frac{E(q^7)^3}{E(q)^4} + 49q\frac{E(q^7)^7}{E(q)^8}. \tag{2}$$