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On certain properties of harmonic numbers

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Abstract

Let H_n be the n-th harmonic number and let u_n be its numerator. For any prime p, let J_p be the set of positive integers n with $p \mid u_n$. In 1991, Eswarathasan and Levine conjectured that J_p is finite for any prime p. It is clear that the p-adic valuation of H_n is not less than $-\lfloor \log_p n \rfloor$. Let T_p be the set of positive integer n such that the p-adic valuation of H_n is equal to $-\lfloor \log_p n \rfloor$. Recently, Carlo Sanna proved that $|J_p \cap [1, x]| \le 129 p^{2/3} x^{0.765}$ and that there exists $S_p \subseteq T_p$ with $\delta(S_p) > 0.273$, where $\delta(X)$ denotes the logarithmic density of the set X of positive integers. He also commented that with his methods $\delta(S_p) > 1/3 - \varepsilon$ cannot be achieved, In this paper, we improve these results. For example, two of our results are: (a) $|J_p \cap [1, x]| \le 3x^{2/3+1/(25\log p)}$; (b) $\delta(T_p)$ exists and $1 - (2\log p)^{-1} \le \delta(T_p) \le 1 - (p \log p)^{-1}$ for all primes $p \ge 13$. In particular, $\delta(T_p) > 0.63$ for all primes p.

Keywords: Harmonic numbers; p-adic valuation; Asymptotic density; Logarithmic density.
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1. Introduction

For any positive integer n, let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

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