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Equidistribution of the crucial measures in non-Archimedean dynamics

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A R T I C L E I N F O

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A B S T R A C T

Text. Let K be a complete, algebraically closed, non-Archimedean valued field, and let $\phi \in K(z)$ with $\deg(\phi) \geq 2$. In this paper we consider the family of functions $\operatorname{ord}\operatorname{Res}_{\phi^n}(x)$, which measure the resultant of ϕ^n at points x in $\mathbf{P}^1_{\mathrm{K}}$, the Berkovich projective line, and show that they converge locally uniformly to the diagonal values of the Arakelov–Green's function $g_{\mu_\phi}(x,x)$ attached to the canonical measure of ϕ. Following this, we are able to prove an equidistribution result for Rumely's crucial measures ν_{ϕ^n}, each of which is a probability measure supported at finitely many points whose weights are determined by dynamical properties of ϕ.

Video. For a video summary of this paper, please visit https://youtu.be/YCCZD1iwe00.

1. Introduction

Let $(K, |\cdot|)$ be a complete, algebraically closed, non-Archimedean valued field, $\mathcal{O}_K$ its ring of integers, and $\mathfrak{m}_K$ its maximal ideal. Denote by k its residue field $k = \mathcal{O}_K/\mathfrak{m}_K$. We normalize the absolute value on K so that $\log_v(|x|) = -\operatorname{ord}_{\mathfrak{m}_K}(x)$.

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This paper is concerned with the dynamics of a rational map $\phi \in K(z)$ of degree $d \geq 2$ on the Berkovich projective line over K, which we denote $\mathbf{P}^1_{\mathrm{K}}$. Rumely has recently introduced two equivariants attached to such a map that carry information about the reduction of conjugates of ϕ [10,11]. The first equivariant is a function $\mathrm{ordRes}_\phi : \mathbf{P}^1_{\mathrm{K}} \to \mathbb{R}$, which measures the resultant of $\mathrm{GL}_2(K)$-conjugates of a homogeneous lift of ϕ; if ϕ has potential good reduction, the locus where this function is minimized identifies the conjugate realizing good reduction.

The second equivariant is a probability measure ν_ϕ called the crucial measure, which is defined as a weighted sum of point masses (see [11] Theorem 6.2):

$$\nu_\phi := \frac{1}{d-1} \sum_{P \in \mathbf{H}^1_{\mathrm{K}}} w_\phi(P) \delta_P \ ;$$

here, $w_\phi(P)$ is a certain weight function which vanishes on points of type I, III and IV and whose values at type II points are determined by the reduction of ϕ at the corresponding point $P \in \mathbf{H}^1_{\mathrm{K}}$; an explicit formula is given in [11] Definition 8. In particular, the weight function is integer valued, and only finitely many points have positive weight. Rumely showed that the formulation of ν_ϕ given above arises naturally when computing the Laplacian of ordRes_ϕ on a canonical subtree $\Gamma_{\widehat{\mathrm{FR}}}$ of $\mathbf{H}^1_{\mathrm{K}}$ (see [11] Corollary 6.5).

These equivariants have been used to establish several important facts in non-Archimedean dynamics. Using the measures ν_ϕ, Rumely has shown that ϕ can have at most $d-1$ repelling type II points in $\mathbf{P}^1_{\mathrm{K}}$ (see [11] Corollary 6.3); prior to this, it was unknown whether there were always finitely many such points. Rumely has also used these measures to show that semistable reduction of ϕ is equivalent to minimality of the resultant (see [11] Theorem 7.4; Szpiro, Tepper and Williams had previously shown that semi-stable reduction *implies* minimality of the resultant using different techniques, and their result holds for morphisms on higher dimension projective spaces; see [14] Theorem 3.3). In [6] Doyle, the author, and Rumely used ordRes_ϕ and ν_ϕ to show that for quadratic rational maps, the points in $\mathrm{supp}(\nu_\phi)$ determine the class of the reduction $\widetilde{\phi}$ in the moduli space $\mathcal{M}_2(k)$.

In this paper, we consider the corresponding equivariants attached to the iterates of ϕ^n. Our first result concerns the functions ordRes_{ϕ^n}:

Theorem 1. *For $x \in \boldsymbol{H}^1_K$, the normalized functions*

$$\frac{1}{d^{2n} - d^n} \,\mathrm{ord}\,\mathrm{Res}_{\phi^n}(x)$$

converge to the diagonal values of the Arakelov–Green's function, $g_\phi(x,x)$ of ϕ. The convergence is locally uniform on $\boldsymbol{H}^1_K$ in the strong topology.

We state and prove a more explicit version of Theorem 1 in Section 3 below (see Theorem 4); in particular, we are able to show that the error term is $O\left(\frac{1}{d^n}\right)$. Our