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Distribution of integral values for the ratio of two linear recurrences

Carlo Sanna

Università degli Studi di Torino, Department of Mathematics, Turin, Italy

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ABSTRACT

Let F and G be linear recurrences over a number field $\mathbb{K}$, and let $\mathfrak{R}$ be a finitely generated subring of $\mathbb{K}$. Furthermore, let $\mathcal{N}$ be the set of positive integers n such that $G(n) \neq 0$ and $F(n)/G(n) \in \mathfrak{R}$. Under mild hypothesis, Corvaja and Zannier proved that $\mathcal{N}$ has zero asymptotic density. We prove that $\#(\mathcal{N} \cap [1, x]) \ll x \cdot (\log\log x / \log x)^h$ for all $x \geq 3$, where h is a positive integer that can be computed in terms of F and G. Assuming the Hardy–Littlewood k-tuple conjecture, our result is optimal except for the term $\log\log x$.

1. Introduction

A sequence of complex numbers $F(n)_{n \in \mathbb{N}}$ is called a *linear recurrence* if there exist some $c_0, \ldots, c_{k-1} \in \mathbb{C}$ $(k \geq 1)$, with $c_0 \neq 0$, such that

$$F(n + k) = \sum_{j=0}^{k-1} c_j F(n + j),$$

E-mail address: carlo.sanna.dev@gmail.com.

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for all $n \in \mathbb{N}$. In turn, this is equivalent to an (unique) expression

$$F(n) = \sum_{i=1}^{r} f_i(n)\, \alpha_i^n,$$

for all $n \in \mathbb{N}$, where $f_1, \dots, f_r \in \mathbb{C}[X]$ are nonzero polynomials and $\alpha_1, \dots, \alpha_r \in \mathbb{C}^*$ are all the distinct roots of the polynomial

$$X^k - c_{k-1}X^{k-1} - \cdots - c_1 X - c_0.$$

Classically, $\alpha_1, \dots, \alpha_r$ and k are called the *roots* and the *order* of F, respectively. Furthermore, F is said to be *nondegenerate* if none the ratios α_i/α_j $(i \neq j)$ is a root of unity, and F is said to be *simple* if all the $f_1, \dots, f_r$ are constant. We refer the reader to [8, Ch. 1–8] for the general theory of linear recurrences.

Hereafter, let F and G be linear recurrences and let $\mathfrak{R}$ be a finitely generated subring of $\mathbb{C}$. Assume also that the roots of F and G together generate a multiplicative torsion-free group. This "torsion-free" hypothesis is not a loss of generality. Indeed, if the group generated by the roots of F and G has torsion order q, then for each $r = 0, 1, \dots, q-1$ the roots of the linear recurrences $F_r(n) = F(qn + r)$ and $G_r(n) = G(qn + r)$ generate a torsion-free group. Therefore, all the results in the following can be extended just by partitioning $\mathbb{N}$ into the arithmetic progressions of modulo q and by studying each pair of linear recurrences F_r, G_r separately. Finally, define the following set of natural numbers

$$\mathcal{N} := \{n \in \mathbb{N} : G(n) \neq 0,\ F(n)/G(n) \in \mathfrak{R}\}.$$

Regarding the condition $G(n) \neq 0$, note that, by the "torsion-free" hypothesis, $G(n)$ is nondegenerate and hence the Skolem–Mahler–Lech Theorem [8, Theorem 2.1] implies that $G(n) = 0$ only for finitely many $n \in \mathbb{N}$. In the sequel, we shall tacitly disregard such integers.

Divisibility properties of linear recurrences have been studied by several authors. A classical result, conjectured by Pisot and proved by van der Poorten, is the Hadamard-quotient Theorem, which states that if $\mathcal{N}$ contains all sufficiently large integers, then F/G is itself a linear recurrence [13,21].

Corvaja and Zannier [7, Theorem 2] gave the following wide extension of the Hadamard-quotient Theorem (see also [6] for a previous weaker result by the same authors).

Theorem 1.1. *If $\mathcal{N}$ is infinite, then there exists a nonzero polynomial $P \in \mathbb{C}[X]$ such that both the sequences $n \mapsto P(n)F(n)/G(n)$ and $n \mapsto G(n)/P(n)$ are linear recurrences.*

The proof of Theorem 1.1 makes use of the Schmidt's Subspace Theorem. We refer the reader to [4] for a survey on several applications of the Schmidt's Subspace Theorem in Number Theory.

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