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Integral isosceles triangle–parallelogram and Heron triangle–rhombus pairs with a common area and common perimeter

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ABSTRACT

In this paper we show that there are infinitely many pairs of integer isosceles triangles and integer parallelograms with a common (integral) area and common perimeter. We also show that there are infinitely many Heron triangles and integer rhombuses with common area and common perimeter. As a corollary, we show there does not exist any Heron triangle and integer square which have a common area and common perimeter.

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1. Introduction

The study of geometrical objects is a very ancient problem. There are many questions in number theory which are related to triangles, rectangles, squares, polygons, and so forth. For example, there is the well-known congruent number problem which asks: given a positive integer n, does there exist a right triangle with rational side lengths whose area is n? As a second example, several researchers have related various types of triangles and quadrilaterals to the theory of elliptic curves. Both Goins and Maddox [5] and Dujella and Peral [4] constructed elliptic curves over $\mathbb{Q}$ coming from Heron triangles. Izadi, Khoshnam, and Moody later generalized their notions to Heron quadrilaterals [7]. In [9] Naskręcki constructed elliptic curves associated to Pythagorian triplets, and Izadi et al. similarly studied curves arising from Brahmagupta quadrilaterals [8].

Another problem connecting geometrical objects with number theory is devoted to the construction of triangles with area, perimeter or side lengths with certain arithmetic properties. Bill Sands asked his colleague R.K. Guy if there were triangles with integer sides associated with rectangles having the same perimeter and area. In 1995, Guy [6] showed that the answer was affirmative, but that there is no non-degenerate right triangle and rectangle pair with the same property. In that same paper, Guy also showed that there are infinitely many such isosceles triangle and rectangle pairs. Several other works in this direction have been solved, all involving pairs of geometric shapes having a common area and common perimeter: two distinct Heron triangles by A. Bremner [1], Heron triangle and rectangle pairs by R.K. Guy and Bremner [2], integer right triangle and parallelogram pairs by Y. Zhang [13], and integer right triangle and rhombus pairs by S. Chern [3].

In this paper we continue this line of study. The first problem we examine regards integer isosceles triangles and integer parallelograms which share a common area and common perimeter. We then consider Heron triangle and integer rhombus pairs. Using the theory of elliptic curves we are able to prove that there are infinitely many examples of each type.

2. Integral isosceles triangle and parallelogram pairs

We first address the case of integral isosceles triangles and parallelograms which have a common (integral) area and common perimeter. As we are requiring the area of the isosceles triangle to be integral, then necessarily the altitude to the non-isosceles side of the triangle must be rational. By the general solution to the Pythagorean equation, we may take the equal legs of the isosceles triangle to have length $m^2 + n^2$, with the base being $2(m^2 - n^2)$ and the altitude $2mn$, for some rational m, n. The perimeter of the triangle is $4m^2$, with an area of $2mn(m^2 - n^2)$. See Fig. 1.

For the parallelogram, we let p, q be the consecutive side lengths, with their intersection angle θ. The perimeter of the parallelogram is $2(p + q)$, while the area is $pq \sin\theta$. In order for the two areas to be equal, then $\sin\theta$ must necessarily be rational. We as-

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