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# Genus-correspondences respecting spinor genus

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ABSTRACT

For two positive definite integral ternary quadratic forms $f$ and $g$ and a positive integer $n$, if $n \cdot g$ is represented by $f$ and $n \cdot dg = df$, then the pair $(f, g)$ is called a *representable pair by scaling* $n$. The set of all representable pairs in $\text{gen}(f) \times \text{gen}(g)$ is called a genus-correspondence. In [6], Jagy conjectured that if $n$ is square free and the number of spinor genera in the genus of $f$ equals to the number of spinor genera in the genus of $g$, then such a genus-correspondence respects spinor genus in the sense that for any representable pairs $(f, g), (f', g')$ by scaling $n$, $f' \in \text{spn}(f)$ if and only if $g' \in \text{spn}(g)$. In this article, we show that by giving a counter example, Jagy's conjecture does not hold. Furthermore, we provide a necessary and sufficient condition for a genus-correspondence to respect spinor genus.



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## 1. Introduction

For a positive definite integral ternary quadratic form

$$f(x,y,z) = ax^2 + by^2 + cz^2 + pyz + qzx + rxy \quad (a,b,c,p,q,r \in \mathbb{Z}),$$

it is quite an old problem determining the set $Q(f)$ of all positive integers $k$ such that $f(x,y,z) = k$ has an integer solution. If the class number of $f$ is one, then one may easily compute the set $Q(f)$ by using, so called, the local–global principle. However, if the class number of $f$ is bigger than 1, then determining the set $Q(f)$ exactly seems to be quite difficult, except some very special ternary quadratic forms. If the integer $k$ is sufficiently large, then the theorem of Duke and Schulze-Pillot in [4] implies that if $k$ is primitively represented by the spinor genus of $f$, then $k$ is represented by $f$ itself.

Recently, W. Jagy proved in [6] that for any square free integer $k$ that is represented by a sum of two integral squares, it is represented by any ternary quadratic form in the spinor genus $x^2 + y^2 + 16kz^2$. To prove this, he introduced, so called *a genus-correspondence*, and proved some interesting properties on the genus-correspondence. To be more precise, let $\text{gen}(f)$ ($\text{spn}(f)$) be the set of genus (spinor genus, respectively) of $f$, for any ternary quadratic form $f$. Let $f$ and $g$ be positive definite integral ternary quadratic forms, and assume that there is a positive integer $n$ such that

$$n \cdot g \text{ is represented by } f \qquad \text{and} \qquad n \cdot dg = df. \tag{1.1}$$

In this article, we denote such a pair $(f, g)$ by a *representable pair by scaling* $n$. Note that $n \cdot f$ is also represented by $g$ for any representable pair $(f, g)$ by scaling $n$. As stated in [6], W.K. Chan proved that for any ternary quadratic form $f' \in \text{gen}(f)$, there is a ternary quadratic form $g' \in \text{gen}(g)$ such that $(f', g')$ is a representable pair by scaling $n$, and conversely for any $\tilde{g} \in \text{gen}(g)$, there is an $\tilde{f} \in \text{gen}(f)$ such that $(\tilde{f}, \tilde{g})$ is also a representable pair by scaling $n$. Jagy defined the set of representable pairs by scaling $n$ by a *genus-correspondence* and proved some properties on a genus-correspondence. He also conjectured that if $n$ is square free and the number of spinor genera in the genus of $f$ equals to the number of spinor genera in the genus of $g$, then such a genus-correspondence *respects spinor genus* in the sense that for any representable pairs $(f, g), (f', g')$ by scaling $n$, where $f' \in \text{gen}(f)$ and $g' \in \text{gen}(g)$,

$$f' \in \text{spn}(f) \qquad \text{if and only if} \qquad g' \in \text{spn}(g). \tag{1.2}$$

In this article, we give an example such that Jagy's conjecture does not hold. In fact, the concept of "genus-correspondence" in [6] is a little bit ambiguous. We modify the notion of a genus-correspondence as follows: For a positive integer $n$, let $\mathfrak{C}$ be a subset of $\text{gen}(f) \times \text{gen}(g)$ such that each pair in $\mathfrak{C}$ is a representable pair by scaling $n$. We say $\mathfrak{C}$ is a genus-correspondence if for any $f' \in \text{gen}(f)$, there is a $g' \in \text{gen}(g)$ such that $(f', g') \in \mathfrak{C}$, and conversely, for any $\tilde{g} \in \text{gen}(g)$, there is an $\tilde{f} \in \text{gen}(f)$ such that $(\tilde{f}, \tilde{g}) \in \mathfrak{C}$. Note that