ARTICLE IN PRESS YJNTH:5756

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

# Eichler–Shimura isomorphism and group cohomology on arithmetic groups ☆,☆☆

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A R T I C L E I N F O

*Article history:*
Received 20 January 2017
Received in revised form 18 April 2017
Accepted 20 April 2017
Available online xxxx
Communicated by D. Goss

*Keywords:*
Eichler–Shimura
Automorphic forms
Cohomology of arithmetic groups
Automorphic representations

A B S T R A C T

In this article, we give a group cohomological interpretation to the Eichler–Shimura isomorphism. For any quaternion algebra $A$ over a totally real field with multiplicative group $G$, we interpret a weight $(k_1, k_2, \cdots, k_d)$-automorphic form of $G$ as a $G(F)$-invariant homomorphism of $(\mathcal{G}_\infty, K_\infty)$-modules. Then the Eichler–Shimura isomorphism is given by the connection morphism provided by the natural exact sequences defining the $(\mathcal{G}_\infty, K_\infty)$-module of discrete series of weight $(k_1, k_2, \cdots, k_d)$.



## 0. Introduction

The Eichler–Shimura isomorphism establishes a bijection between the space of modular forms and certain cohomology groups with coefficients in a space of polynomials. More precisely, let $k \geq 2$ be an integer and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, then we have the following isomorphism of Hecke modules

☆ The author is supported in part by DGICYT Grant MTM2015-63829-P.

☆☆ The author/authors have been partially funded by the CERCA Programme of the Generalitat de Catalunya.

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http://dx.doi.org/10.1016/j.jnt.2017.04.007

$$M_k(\Gamma,\mathbb{C})\oplus S_k(\Gamma,\mathbb{C})\simeq H^1(\Gamma,V(k)^\vee),\tag{0.1}$$

where $V(k)^\vee$ is the dual of the $\mathbb{C}$-vector space of homogeneous polynomials of degree $k-2$, $M_k(\Gamma,\mathbb{C})$ is the space of modular forms of weight $k$ and $S_k(\Gamma,\mathbb{C})\subset M_k(\Gamma,\mathbb{C})$ is the subspace of cuspidal modular forms (see [6, Thm. 8.4] and [4, Thm. 6.3.4]).

This isomorphism can be interpreted in geometric terms. Indeed, a modular form of weight $k$ can be interpreted as a section of certain sheaf of differential forms on the open modular curve attached to $\Gamma$. With this in mind, the Eichler–Shimura isomorphism can be obtained comparing deRham and singular cohomology, noticing that the singular cohomology of the open modular curve is given by the group cohomology $H^\bullet(\Gamma,V(k)^\vee)$. The aim of this paper is to omit this geometric interpretation and to provide a new group cohomological interpretation.

Let us remark that the identification (0.1) provides an integral and rational structure to the space of modular forms, since the space of polynomials $V(k)$ has integral and rational models, namely, the space of polynomials with integer and rational coefficients.

The restriction of the Eichler–Shimura isomorphism to the spaces of cuspidal modular forms is given by the morphisms

$$\partial^\pm: S_k(\Gamma,\mathbb{C})\longrightarrow H^1(\Gamma,V(k)^\vee),$$

where

$$\partial^\pm(f)(\gamma)(P)=\int_{z_0}^{\gamma z_0}P(1,-\tau)f(\tau)d\tau\pm\int_{z_0}^{\gamma z_0}P(1,\bar\tau)f(-\bar\tau)d(-\bar\tau),$$

for any $z_0$ in $\mathcal{H}$ the Poincaré hyperplane, $\gamma\in\Gamma$ and $P\in V(k)$. In fact, the morphism defining the cuspidal part of (0.1) is given by

$$\begin{array}{ccc} S_k(\Gamma,\mathbb{C})\oplus\overline{S_k(\Gamma,\mathbb{C})} & \longrightarrow & H^1(\Gamma,V(k)^\vee)\\ (f_1,\bar f_2) & \longmapsto & (\partial^++\partial^-)(f_1)+(\partial^+-\partial^-)(f_2). \end{array}$$

The image of $\partial^\pm$ lies in the subspaces $H^1(\Gamma,V(k)^\vee)^\pm\subset H^1(\Gamma,V(k)^\vee)$ where the natural action of $\omega=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ normalizing $\Gamma$ acts by $\pm1$.

In a general setting, $F$ is a totally real number field of degree $d$, $G$ is the multiplicative group of a quaternion algebra $A$ over $F$, and $\phi$ is a weight $\underline{k}=(k_1,\cdots,k_d)$ cuspidal automorphic form of $G$ with level $\mathcal{U}\subset G(\mathbb{A}^\infty)$ and central character $\psi:\mathbb{A}^\times/F^\times\to\mathbb{C}^\times$. We assume that $\psi_{\sigma_i}(x)=\mathrm{sign}(x)^{k_i}|x|^{\mu_i}$, for any archimedean place $\sigma_i:F\hookrightarrow\mathbb{R}$. In this scenario, by an automorphic form we mean a function on $\mathcal{H}^r\times G(\mathbb{A}^\infty)/\mathcal{U}$, where $\mathcal{H}$ is the Poincaré upperplane and $r$ is the cardinal of the set $\Sigma$ of archimedean places where $A$ splits, with values in $\bigotimes_{\sigma_i\in\infty\setminus\Sigma}V_{\mu_i}(k_i)^\vee$ (see §1.2 for a precise definition of $V_{\mu_i}(k_i)$), that satisfies the usual transformation laws with respect to the weight-$k_i$-actions of $G(F)$. The interesting cohomology subgroups to consider are: