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Visible lattice points and the Extended Lindelöf Hypothesis

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ABSTRACT

We consider the number of visible lattice points under the assumption of the Extended Lindelöf Hypothesis. We get a relation between visible lattice points and the Extended Lindelöf Hypothesis. And we also get a relation between visible lattice points over $\mathbf{Q}(\sqrt{-1})$ and the Gauss Circle Problem.

1. Introduction

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. We consider an m-tuple of ideals $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ of $\mathcal{O}_K$ as lattice points in $Frac(\mathcal{O}_K)^m$. When $\mathcal{O}_K = \mathbf{Z}$ they are ordinally lattice points. We say that a lattice point $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ is visible from the origin, if $\mathfrak{a}_1 + \cdots + \mathfrak{a}_m = \mathcal{O}_K$.

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There are many results about the number of visible lattice point from 1800s. In the case $K = \mathbf{Q}$, D.N. Lehmer proved that the density of the set of visible lattice points in $\mathbf{Q}^m$ is $1/\zeta(m)$ in 1900 [7]. And in general case, B.D. Sittinger proved the number of visible lattice points $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ in K^m with $\mathfrak{N}\mathfrak{a}_i \leq x$ for all $i = 1, \ldots, m$ is

$$\frac{c^m}{\zeta_K(m)} x^m + (\text{Error term}),$$

where ζ_K is the Dedekind zeta function over K and c is a positive constant depending only on K [9].

Let $V_m(x, K)$ denote the number of visible lattice points $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ with $\mathfrak{N}\mathfrak{a}_i \leq x$ for all $i = 1, \ldots, m$. When $K = \mathbf{Q}$, $V_m(x, \mathbf{Q})$ means the number of visible lattice points in $(0, x]^m$. And we let $E_m(x, K)$ denote its error term, i.e. $E_m(x, K) = V_m(x, K) - (cx)^m/\zeta_K(m)$.

In the case $K = \mathbf{Q}$, we proved that the exact order of $E_m(x, \mathbf{Q})$ is x^{m-1} for $m \geq 3$ [10]. But we do not know about the exact order of $E_m(x, K)$. In this paper, we consider better upper order of $E_m(x, K)$ under the situation that the Extended Lindelöf Hypothesis is true. The statement of our main theorem is the following.

Theorem. *If we assume the Extended Lindelöf Hypothesis, we get*

$$E_m(x, K) = O(x^{m-1/2+\varepsilon})$$

for all algebraic number field K and for all $\varepsilon > 0$.

As a result, we can think of considering the number of visible lattice points as considering the Extended Lindelöf Hypothesis. And we show that the number of visible lattice points in $\mathbf{Q}(\sqrt{-1})^m$ are associated with the Gauss Circle Problem.

2. The Extended Lindelöf Hypothesis

The Dedekind zeta function ζ_K over K is considered as a generalization of the Riemann zeta function and ζ_K is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}\mathfrak{a}^s},$$

with the sum taken over all nonzero ideals of $\mathcal{O}_K$.

Riemann proposed that all non-trivial zeros of the Riemann zeta function is on the line $\Re(s) = 1/2$ in his paper [8]. The Extended Riemann Hypothesis over algebraic number field is known as a generalization of the Riemann hypothesis. The statement of the Extended Riemann Hypothesis is "for all algebraic number field K all non-trivial zeros of the Dedekind zeta function is on the line $\Re(s) = 1/2$".

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