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Some relations on Fourier coefficients of degree 2 Siegel forms of arbitrary level

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ABSTRACT

We extend some recent work of D. McCarthy, proving relations among some Fourier coefficients of a degree 2 Siegel modular form F with arbitrary level and character, provided there are some primes p so that F is an eigenform for the Hecke operators $T(p)$ and $T_1(p^2)$.

1. Introduction

In a recent paper [3], McCarthy derives some nice results for Fourier coefficients and Hecke eigenvalues of degree 2 Siegel modular forms of level 1, extending some classical results regarding elliptic modular forms. In particular, with F a degree 2, level 1 Siegel modular form that is an eigenform for all the Hecke operators $T(p)$, $T(p^2)$ (p prime), and $a(T)$ denoting the Tth Fourier coefficient of F, McCarthy shows that:

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(a) provided that $a(I) = 1$ and p is prime, the $T(p)$-eigenvalue $\lambda(p)$ and the $T(p^2)$-eigenvalue $\lambda(p^2)$ are described explicitly in terms of $a(pI)$ and $a(p^2I)$;
(b) for $r \geq 1$, $a(I)a(p^{r+1}I)$ is described explicitly in terms of $a(I)$, $a(pI)$, $a(p^{r-1}I)$, $a\begin{pmatrix} p^{r-1} & \\ & p^{r+1} \end{pmatrix}$, and $a\left(p^r\begin{pmatrix} (1+u^2)/p & u \\ u & p \end{pmatrix}\right)$ where $1 \leq u < p/2$ with $u^2 \not\equiv 1\ (p)$;
(c) if $a(I) = 0$ then $a(mI) = 0$ for all $m \in \mathbb{Z}_+$; further, if $m, n \in \mathbb{Z}_+$ with $(m, n) = 1$, then $a(I)a(mnI) = a(mI)a(nI)$.

(As defined in Sec. 2, $T_2(p^2)$ is the Hecke operator associated with the matrix $\mathrm{diag}(p, p, 1/p, 1/p)$, $T_1(p^2)$ is the Hecke operator associated with the matrix $\mathrm{diag}(p, 1, 1/p, 1)$, and $T(p^2) = T_2(p^2) + p^{k-3}T_1(p^2) + p^{2k-6}$. In [2], for $\chi = 1$, $T(p^2)$ is denoted by $\widetilde{T}_2(p^2)$.) McCarthy's approach begins with some formulas from [1], which are somewhat cumbersome.

In this note we use the formulas from [2] that give the action of Hecke operators on Fourier coefficients of a Siegel modular form F, allowing for arbitrary level and character, and giving a simpler proof of McCarthy's above results (with no restriction on the level or character). Here when we say that a modular form has weight k, level $\mathcal{N}$ and character χ, we mean that it transforms with weight k and character χ under the congruence subgroup

$$\Gamma_0(\mathcal{N}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbb{Z}) :\ \mathcal{N}|C \right\},$$

where $Sp_2(\mathbb{Z})$ is the symplectic group of 4×4 integral matrices. We work with "Fourier coefficients" attached to lattices (as explained below), making it simpler to work with the image of F under a Hecke operator. For p prime and degree 2, the local Hecke algebra is generated by $T(p)$, $T_1(p^2)$ and $T_2(p^2)$. When $\mathcal{N} = 1$, Proposition 5.1 of [2] gives a relation between these generators, from which we deduce that with $p \nmid \mathcal{N}$, $T(p)$ and $T_1(p^2)$ generate the local Hecke algebra, as do $T(p)$ and $\widetilde{T}_2(p^2)$. However, when $p|\mathcal{N}$, we have $T_2(p^2) = (T(p))^2$. Hence in this note we use the local generators $T(p)$ and $T_1(p^2)$; to more easily apply the results of [2], we use the operator

$$\widetilde{T}_1(p^2) = T_1(p^2) + \chi(p)p^{k-3}(p+1)$$

in place of $T_1(p^2)$.

Using some rather special aspects of working with degree 2 Siegel modular forms, we prove the following extensions of [3].

Theorem 1.1. *Suppose that F is a degree 2 Siegel modular form of weight $k \in \mathbb{Z}_+$, level $\mathcal{N}$ and character χ with Fourier expansion*

$$F(\tau) = \sum_T a(T)\exp(2\pi i Tr(T\tau)).$$

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