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Generating weights for the Weil representation attached to an even order cyclic quadratic module ☆

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ABSTRACT

Text. We develop geometric methods to study the generating weights of free modules of vector-valued modular forms of half-integral weight, taking values in a complex representation of the metaplectic group. We then compute the generating weights for modular forms taking values in the Weil representation attached to cyclic quadratic modules of order $2p^r$, where $p \geq 5$ is a prime. We also show that the generating weights approach a simple limiting distribution as p grows, or as r grows and p remains fixed.

Video. For a video summary of this paper, please visit https://youtu.be/QNbPSXXKot4.

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Imaginary quadratic field
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Theta function

1. Introduction

The aim of this paper is to compute the generating weights of modules of modular forms associated to Weil representations attached to finite quadratic modules of order $2p^r$. Our approach is to extend the main geometric results developed in [CF17] so that they can be applied in the half-integral weight setting of the present paper. In the half-integral weight setting, one can replace the weighted projective line $\mathbf{P}(4,6)$ of [CF17] with $\mathbf{P}(8,12)$, and thereby recover the results of [CF17] with natural modifications. The modifications are explained in Section 2.

The computation of generating weights is a fundamental problem in the theory of vector-valued modular forms (see [Mar11,CF17,FM17]), which is equivalent to determining how certain vector bundles decompose into line bundles [CF17]. For unitary representations of $\mathrm{SL}_2(\mathbf{Z})$, the only obstacle to such problems is presented by the forms of weight one, while for nonunitary representations, there could be a greater number of problematic weights (although always finitely many by [Mas07] and [CF17]). In the half-integral weight setting, the authors had believed that—even if one only considers unitary representations of $\mathrm{Mp}_2(\mathbf{Z})$—the multiplicities of the three critical weights $1/2$, 1 and $3/2$ would all be difficult to compute. It turns out that one can use results of Skoruppa [Sko85,Sko08] and Serre–Stark [SS77], along with Serre duality, to handle weights $1/2$ and $3/2$ in many cases, and eliminate weight one by imposing parity restrictions (analogous to restricting to representations of $\mathrm{PSL}_2(\mathbf{Z})$ versus odd representations of $\mathrm{SL}_2(\mathbf{Z})$). The general class of Weil representations that we consider in this paper is introduced in Section 3 below. Section 4 specializes to certain cyclic Weil representations where it is possible to compute the generating weights of the corresponding module of vector-valued modular forms using the results discussed above. The result of these computations can be found in Table 1 of Section 4 below.

The most interesting part of these computations turns out to be the evaluation of $\mathrm{Tr}(L)$, where L denotes a so-called *exponent matrix* for a representation ρ of $\mathrm{Mp}_2(\mathbf{Z})$. Such a matrix L satisfies $\rho(T) = e^{2\pi i L}$, where T denotes a lift of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ to $\mathrm{Mp}_2(\mathbf{Z})$. In Section 4, we consider the finite quadratic module $\mathbf{Z}/2p^r\mathbf{Z}$ with the quadratic form $q(x) = \frac{1}{4p^r}x^2$, where $p \geq 5$ is an odd prime. If ρ is the corresponding Weil representation, then one finds that

$$\mathrm{Tr}(L) = \sum_{x=1}^{2p^r} \left\{ -\frac{x^2}{4p} \right\},$$

where $\{x\} \in [0,1)$ denotes the fractional part of a real number x. Theorem 4.5 below shows that $\mathrm{Tr}(L)$ is asymptotic to $p^r = \frac{1}{2}\dim\rho$, with a correction term of order