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On generalized trigonometric functions and series of rational functions

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A B S T R A C T

Here we introduce a way to construct generalized trigonometric functions associated with any complex polynomials. The well known trigonometric functions can be seen to be associated with polynomial $x^2 - 1$. We will show that those generalized trigonometric functions have algebraic identities which generalizes the well known $\sin^2(x) + \cos^2(x) = 1$. One application of the generalized trigonometric functions is in evaluating infinite series of rational functions.

1. Introduction

Trigonometric functions are very commonly used in mathematics. The Euler's identity tells that for all $x \in \mathbb{C}$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \text{ and } \cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

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We see that the trigonometric functions $\sin(x)$ and $\cos(x)$ are just certain linear combinations of exponential functions. From here it is a natural question to ask whether we can find other linear combinations of exponential functions to obtain some functions that share some properties with trigonometric functions such as addition formulae and algebraic identities. We observe that in the above formulae those exponential functions have exponents $ix, -ix$. The factors $\pm i$ are roots of polynomial $x^2 + 1 = 0$ in $\mathbb{C}$. This motivates the following construction:

Let $P \in \mathbb{C}[x]$ be a polynomial of degree $m \geq 1$. Then let $r_1, r_2, \dots, r_m$ be the roots of P. We shall construct m functions

$$S_l^P : \mathbb{C} \to \mathbb{C}, l \in \{0, 1, 2, \dots, m-1\}.$$

The functions have form $S_l^P(x) = \sum_{j=1}^m T_{l,j}^P e^{-ir_j x}$ with $T_{l,j}^P \in \mathbb{C}$.

Now we are going to describe the coefficients $T_{l,j}^P$. For any choice of l different numbers say $s_1, \dots, s_l$ from $1, 2 \dots m-1$ we can assign the product

$$r_{s_1} r_{s_2} \dots r_{s_l},$$

and we can call such a product an *unordered l-tuple*. For example if P has 5 roots, then all the unordered 3-tuples are

$$r_1r_2r_3, r_1r_2r_4, r_1r_2r_5, r_2r_3r_4, r_2r_3r_5, r_3r_4r_5, r_1r_3r_4, r_1r_3r_5, r_1r_4r_5, r_2r_4r_5.$$

Further more we can say *l-tuple with j* as a unordered l-tuple with index j, for example if P has 5 roots as above we have all 3-tuples with 2 are

$$r_1r_2r_3, r_1r_2r_4, r_1r_2r_5, r_2r_3r_4, r_2r_3r_5, r_2r_4r_5.$$

Now we can define

$$T_{l,j}^P = \sum_{a \in \{\text{All l-tuple with j}\}} a,$$

where 0-tuples are defined to be the coefficient of the highest term of polynomial P. Later we can just write the sum without explicitly writing down the terms and there is no confusion.

$$T_{l,j}^P = \sum_{\text{All l-tuple with j}}.$$

For example, when P has 5 roots, then

$$T_{3,2}^P = r_1r_2r_3 + r_1r_2r_4 + r_1r_2r_5 + r_2r_3r_4 + r_2r_3r_5 + r_2r_4r_5.$$

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