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The Roman harmonic numbers revisited

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ABSTRACT

Two decades ago, Steven Roman, Daniel E. Loeb and Gian-Carlo Rota introduced a family of harmonic numbers in their study of harmonic logarithms. We propose to refer to those numbers as *Roman harmonic numbers*. With the purpose of revitalizing the study of these mathematical objects, we recall here their known properties and unveil additional ones. An integral representation, several generating relations, and a collection of sum rules involving those numbers are presented. It is also shown that higher derivatives of the Pochhammer and reciprocal Pochhammer symbols are easily expressed in terms of Roman harmonic numbers.

1. Introduction

In the course of a research on the so called epsilon expansions of Appell and Kampé de Fériet functions [11], we considered convenient to define what we called modified generalized harmonic numbers, $\hat{H}_n^{(k)}$, in the form

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$$\hat{H}_0^{(k)} = \delta_{k,0}\,, \qquad \hat{H}_n^{(k)} \equiv \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j^k}\,, \quad n \geq 1\,. \tag{1}$$

Later on, we realized that these mathematical objects, with the name of harmonic numbers and denoted by $c_n^{(k)}$, had already been introduced, by Loeb and Rota [16] and by Roman [22,23], to give explicit expressions of the harmonic logarithms. On the other hand, the moments

$$\bar{d^p} = \int_0^\infty d\delta\, \delta^p\, \mathcal{P}(\delta) \qquad p = 0, 1, 2, \ldots\,, \tag{2}$$

of the quantum probability distribution

$$\mathcal{P}(\delta) = 2\,(D_\varepsilon - 1)\; e^{-2\delta}\left(1 - e^{-2\delta}\right)^{D_\varepsilon - 2}\,, \quad D_\varepsilon = 2, 3, \ldots\,, \tag{3}$$

discussed by Coffey [7] and used (for $p = 1$ and 2) to approximately calculate the decoherence factors for D_ε-dimensional quantum states, are closely related to the same harmonic numbers, as shown below. In spite of their interesting properties, the $c_n^{(k)}$ have not received, in our opinion, due attention by number theorists. Here we intend to arouse the interest in those harmonic numbers by recalling their known properties and presenting some others.

Several generalizations of harmonic numbers have been used by different authors [2–6,10,14,15,24]. To avoid misunderstandings, we adopt the name of *Roman harmonic numbers* for those introduced in Refs. [16,22], and [23]. One of the possible equivalent definitions of these numbers is given in Section 2, which recalls also their main properties, found by Roman [22,23] and by Loeb and Rota [16]. Section 3 shows, firstly, the connection of those numbers with known nested sums. Their integral representation allows to discover the relation between Roman harmonic numbers and Coffey's quantum distribution moments. Then, several generating relations, sum rules, and additional properties not mentioned before are presented. As an application, expressions are given of the derivatives of the Pochhammer and reciprocal Pochhammer symbols in terms of the $c_n^{(k)}$. To end, some pertinent comments are included in Section 4.

2. First definitions and known properties

Before recalling the Roman harmonic numbers, some auxiliary definitions, given in [16, 22], and [23], are necessary. We present them with the notation used in these references.

Definition 1. (Roman number.) The *Roman n* is defined to be

$$\lfloor n \rceil = \begin{cases} n & \text{for} \quad n \neq 0\,, \\ 1 & \text{for} \quad n = 0\,. \end{cases} \tag{4}$$