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Proof of some divisibility results on sums involving binomial coefficients

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Abstract. By using the Rodriguez-Villegas-Mortenson supercongruences, we prove four supercongruences on sums involving binomial coefficients, which were originally conjectured by Sun. We also confirm a related conjecture of Guo on integer-valued polynomials.

Keywords: Supercongruences; Delannoy number; Legendre symbol; Zeilberger algorithm

MR Subject Classifications: Primary 11A07; Secondary 33C05

1 Introduction

In 2003, Rodriguez-Villegas [11] conjectured 22 supercongruences for hypergeometric Calabi-Yau manifolds of dimension $d \le 3$. For manifolds of dimension $d = 1$, associated to certain elliptic curves, four conjectural supercongruences were posed. Mortenson [8, 9] first proved these four supercongruences by using the Gross-Koblitz formula.

Theorem 1.1 *(Rodriguez-Villegas-Mortenson) Suppose $p \ge 5$ is a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(1/2)_k^2}{(1)_k^2}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\equiv\left(\frac{-3}{p}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{(1/4)_k(3/4)_k}{(1)_k^2}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{(1/6)_k(5/6)_k}{(1)_k^2}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol and $(x)_k = x(x+1)\cdots(x+k-1)$.

Sun [12] introduced the following two kinds of polynomials:

$$d_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}2^k \quad\text{and}\quad s_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\binom{x+k}{k}.$$

Note that $d_n(m)$ are the Delannoy numbers, which count the number of paths from $(0, 0)$ to (m, n), only using steps $(1, 0), (0, 1)$ and $(1, 1)$. For more information on Delannoy numbers, one can refer to [2].

The first aim of this paper is to prove the following result, which was originally conjectured by Sun [12, Conjecture 6.11].